THE TEACHING OF MATHEMATICS

REVISED AND ENLARGED EDITION

KULBIR SINGH SIDHU

The invasion of computers in every walk of life in the near future makes it significant that mathematics is understood by all. Moreover, it is a stimulator of clear and logical thinking rather that a dull subject.

In this work, the author has attempted to provide an understanding of certain basic concepts of teaching elementary mathematics and their application to problem-solving. The discussion of the new teaching techniques—with particular reference to the conditions in Indian schools—is a special feature of the book. Each chapter provides a stimulating exercise to test the students' grasp of the subject. This comprehensive book should be very useful to the students and teachers of education.

Rs 100

4348
21.4.89

THE TEACHING OF MATHEMATICS

/c

The Teaching of Mathematics

KULBIR SINGH SIDHU
M.A. (Maths. Psy.) M.Ed. (Gold Medallist) Ph.D., P.E.S.
Senior Lecturer
Govt. College of Education,
Jalandhar

S.C.E.R.T., West Bengal
Library
Calcutta
25/3, B. C. Road

STERLING PUBLISHERS PRIVATE LIMITED
NEW-DELHI-110016
BANGALORE-560001

STERLING PUBLISHERS PRIVATE LIMITED
L-10, Green Park Extension, New Delhi-110016
24, Race Course Road, Bangalore-560001

21-4-89

C.E.R.T., West Bengal
ate 21-4-89
ec No 4348

THE TEACHING OF MATHEMATICS

© 1986, *Kulbir Singh Sidhu*

372.7
SID

First Edition 1967
Reprinted 1969
Second Revised Edition 1971
Reprinted 1973
Reprinted 1974
Reprinted 1975
Reprinted 1976
Reprinted 1977
Reprinted 1978
✓ Third Revised Edition 1981
Reprinted 1982
Reprinted 1984
Reprinted 1985
Reprinted 1986

PRINTED IN INDIA

Published by S.K. Ghai, Managing Director,
Sterling Publishers Private Limited, L-10, Green Park Extension,
New Delhi-110016 and printed at Ram Printograph (India), New Delhi-110016

PREFACE TO THE THIRD EDITION

WITH GOD'S GRACE the author has at last been able to discharge his duty of revising the book. He is conscious that this revision was long over due. He could not carry it out earlier due to unavoidable circumstances. He promises to revise it more frequently in future.

The author is full of gratitude to the readers for the appreciation shown by them to his book. He hopes, this third revised edition will provide greater satisfaction to them. It has been prepared to give comprehensive treatment to the syllabi of this subject as prescribed in various universities. Quite a number of new topics have been brought in and some of the out-dated and irrelevant topics have been dropped. The new edition is presented to the readers with the hope that it will substantially meet their requirements and will be of greater use to them.

KULBIR SINGH SIDHU

PREFACE TO THE FIRST EDITION

THE ATOM HAS been conquered and outer space penetrated. It is mathematics that has pushed and is pushing forward the frontiers of scientific and technical knowledge, discovery and invention. The subject has also rich potentialities of affording true enjoyment to its students.

The teaching of mathematics presents numerous day to day problems for the teacher of mathematics. Due to inefficient and stereotyped methods of teaching, the subject is sometimes decried as dull and dry.

The author has for many years taught secondary school mathematics and methods of teaching the subject to training classes. In writing this book he has attempted to simplify and even solve typical problems of teaching of mathematics, to cross-fertilize theory and practice and to motivate mathematics teachers and students to pursue the study of the subject—to rouse consciousness of the subject. He has discussed the new teaching techniques with particular reference to the conditions obtaining in Indian schools. Difficult steps in the teaching process have been analysed and elucidated on lines found by the author and his colleagues to be successful in everyday classroom situations.

It is hoped that this book will cover the syllabi for the B.T., B. Ed., L.T. classes of most of the Indian universities.

The western literature on the subject though of fundamental nature, does not always meet the requirements of field workers in India. So the necessary material has been given Indian background and colour.

Help has been taken from standard mathematical works of various countries. The author acknowledges his debt to their writers.

The book should meet the changed requirements of the new higher secondary school syllabi, even in the new areas such as the teaching of Advanced Algebra, Trigonometry and Solid Geometry.

Any suggestions for the improvement of this book will be gratefully acknowledged and adopted.

15 June 1967

KULBIR SINGH SIDHU

CONTENTS

Preface to the Third Edition v

Preface to the First Edition vi

CHAPTER ONE

The Nature of Mathematics 1

Science of Logical Reasoning; Mathematical Language and Symbolism; Pure and Applied Mathematics; Euclidean and Non-Euclidean Geometry; Modern Mathematics.

CHAPTER TWO

The Educational Values of Mathematics 13

Educational Values; Values and Aims; Practical Value; Disciplinary Value; Cultural Value; Development of Concentration; Art of economical living; Power of Expression; Self-reliance; Attitude of Discovery; Understanding of Popular Literature; Quality of hard work; Aims of Teaching Mathematics; The place of Mathematics in every-day life; The place of Mathematics in the school curriculum.

CHAPTER THREE

History of Mathematics 26

Value of History; the Babylonians; The Egyptians; The Greeks; The Ionic School; The School of Pythagoreans; The Sophist School; The Platonic School; The First Alexandrian School; The Second Alexandrian School; The Romans; The Chinese; The Japanese; The Hindus; The Arabs. Historical reviews of developments; Notation system Weights and measures; Logarithms; Computer mathematics.

CHAPTER FOUR

Some Great Mathematicians 50

Aryabhata; Brahmagupta; Bhaskara; S. Ramanujan; Euclid; Pythagoras; Gauss.

CHAPTER FIVE

The Relation of Mathematics with other subjects 65

Mathematics and Physics; Mathematics and Chemistry; Mathematics and Biology; Mathematics and Engineering; Mathematics and Agriculture; Mathematics and Economics; Mathematics and Psychology; Mathematics and Logic; Mathematics and Philosophy; Mathematics and Fine Arts.

CHAPTER SIX

Curriculum in Mathematics 78

The Dynamic Approach to Curriculum. The Teacher's Concern; Objectives of Curriculum; As a means to an end; Flexibility; Why revision; Who should organise and revise; Principles of construction; Criterion of Practical Value; Criterion of Preparatory Value; Criterion of Disciplinary Value; Criterion of Cultural Value; Use is the Chief Criterion; Child-centric Curriculum; A Comprehensive Curriculum; Psychological and Logical Arrangement; Scope for Practical Work; The Criterion of Difficulty; Voice of the Teacher; Topical versus Spiral; The Principle of Cyclic Order; Incidental versus Organised Teaching; Individual versus Class Teaching; The Dalton Plan; The Project Plan; From the Empirical to the Rational; The Principle of Correlation; Problems.

CHAPTER SEVEN

Methods of Teaching 102

Lecture; Dogmatic; Inductive-Deductive; Heuristic; Analytic; Synthetic; Laboratory; Project; Topical; Concentric; Problem; The Best Method.

CHAPTER EIGHT

Aids to Teaching and Mathematics Laboratory 138

Personal equipment of the student; Equipment of the mathematics laboratory; Placement of the material; Blackboard; Concrete Material; Charts; Models; Excursions; Collections; Filmstrips; Radio; Some special instruments; Home-made equipment.

CHAPTER NINE

Mathematics Library and the Text-Book 148

Importance of Library; General School Library; Mathematics Department Library; The use of Library; The Text-Book; Its Importance; How to use the Text-Book; Essentials of a Good Text-Book.

CHAPTER TEN

Defects in the Present-Day Teaching of Mathematics in Schools and their Possible Remedies 156

Teacher's qualification; Teacher's burden; Teacher's salary; Teacher's attitude; Lack of purpose; Lack of equipment; Method of teaching; Rigour in study; Large classes; Practical aspect; Mathematical language; Syllabus; Text-books; The students; Child-centric approach; Libraries and Laboratories; Ban on short-cut methods; Examinations.

CHAPTER ELEVEN

Arousing and Maintaining Interest in Mathematics—A list of Study Suggestions 162

Principles of Meaningful Learning; Mathematics Club; Intellectual activity; Applications to other fields of study; Application to professional fields; Practical values; Cultural values; Recreational values; Practical work; The Principle of change; Physical conditions for study; Psychological conditions for study; A list of study suggestions; Principles of Meaningful Learning; Mathematics Club.

CHAPTER TWELVE

Gifted Students and Backward Students 174

Gifted students; Locating the gifted; Enrichment programme for the gifted; Backward students; General backwardness; Particular backwardness; Causes and remedies

CHAPTER THIRTEEN

Techniques of Teaching 180

Oral work; Drillwork; Homework; Assignment; Selfstudy; Group work Review; Supervised Study

CHAPTER FOURTEEN

Mathematics Teacher 194

Pre-requisite qualifications; Professional training; Selective academic training; Supervised practice teaching; In-service training; Professional activities; School activities; Mathematical organisations; Departmental duties; Administrative duties; Community activities.

CHAPTER FIFTEEN

Measurements and Evaluation 201

Purposes of Evaluation; Criteria for measuring tests; Validity; Reliability; Administrability; Student consciousness; Motivation for the student; Utility, Prognosis and Diagnosis; Survey Test; Achievement Test; New Type Tests; Classification of new type tests.

CHAPTER SIXTEEN

The Teaching of Arithmetic 210

What is Arithmetic; Aims of teaching Arithmetic; Methods of teaching; Teaching of Numeration and Notation; Four simple rules; L.C.M. and G.C.M.; Fractions; Decimal Fractions; Metric Measures, Square Root; Practice; Unitary Method; Ratio; Proportion; Partnership; Percentage; Profit and Loss; Average; Interest, Simple and Compound; Discount; Time and Work; Area; Volume; Stocks and Shares.

CHAPTER SEVENTEEN

The Teaching of Household Accounts 260

How to Teach Household Accounts

CHAPTER EIGHTEEN

The Teaching of Algebra 264

What is Algebra; Why is Algebra taught; Teaching of the Signed Numbers; Brackets; First degree equations; Graphs; Factorising and Formulae; Indices; Quadratic Equations.

CHAPTER NINETEEN

The Teaching of Geometry 288

Geometry; Demonstrative Geometry; Practical Geometry; Why teach Geometry; Its functions in the Middle School; Its Functions in the High School; Its Functions in the Senior High School; Different stages of Teaching Geometry; When to introduce reasoning; Some more teaching points; Axioms and Postulates; Some Fundamental concepts in Geometry; When and How should definitions be taught; How to teach Propositions; Kinds of proofs; Converse, Contrapositive Reciprocal and Inverse Theorems; The Key Propositions; Symbolic Notation; Figure; Simple directions for learning Theorems; Difference between Teaching Euclid and Euclid Teaching; Induction and Deduction; Synthesis and Analysis; Angle-Sum of a Triangle; Congruent Triangles; Similarity; Locus; Riders; How should the riders be attempted; Helpful methods in solving exercises; Constructions; Directions for Construction Work; Analysis in Construction Work.

CHAPTER TWENTY

The Teaching of Trigonometry 3 32

Why to teach it; Which parts of Trigonometry should be taught; Developing the Meaning of a Trigonometric Functions; Inverse Functions; Teaching the Functions of Special Angles; Functions of the General Angle; Teaching the use of Trigonometric Functions; Teaching the Functions of Two Angles; Variation of the Functions; The Solution of Triangles; Teaching Radian Measure; Teaching Trigonometric Equations and Identities.

CHAPTER TWENTY-ONE

The Teaching of Solid Geometry 351

What is Solid Geometry; Its Introduction; Some of the Axioms Scope of Solid Geometry; Methods to Teach; Parallel Planes; Projection; Scope of Correlation.

CHAPTER TWENTY-TWO

Lesson Plans 360

(*i*) Proposition : A side of a triangle is produced, so as to form an exterior angle. Prove that the sum of the opposite $\angle$ s is equal to this exterior angle.

(*ii*) Identity $(a^2-b^2)=(a+b)(a-b)$.

(*iii*) Volume of a Cuboid.

(*iv*) Compound Interest.

(*v*) Sin (A+B)=Sin A Cos B+Cos A Sin B

Bibliography 381

CHAPTER ONE

THE NATURE OF MATHEMATICS

ACCORDING TO VARIOUS definitions, mathematics is the science of measurement, quality and magnitude. According to *New English Dictionary*—"Mathematics—in a strict sense—is the abstract science which investigates deductively the conclusions implicit in the elementary conceptions of spatial and numerical relations." It has also been defined as the science of number and space. Its Hindi or Punjabi name is 'Ganita' which means the science of calculation. It is a systematised, organised and exact branch of science.

The term 'Mathematics' has been interpreted and explained in various ways. It is the numerical and calculation part of man's life and knowledge. It helps the man to give exact interpretation to his ideas and conclusions. It deals with quantitative facts and relationships as well as with problems involving space and form. It also deals with relationship between magnitudes. It enables the man to study various phenomenon in space and establish various relationships between them. It explains that this science is a by-product of our empirical knowledge. From our observations of physical and social environment, we form certain intuitive ideas or notions called postulates and axioms. By a process of reasoning, we move upwards and work out mathematical results at the abstract level. "Mathematics may also be defined as the science of abstract form. The discernment of structure is essential no less to the appreciation of a painting or a symphony than to understand the behaviour of a physical system; no less in economics than in astronomy. Mathematics studies order abstracted from the particular objects and phenomena which exhibit it and in a generalised form."

SCIENCE OF LOGICAL REASONING

Mathematics is also called the science of logical reasoning. In it, we approach everything with a question mark in our mind. As Locke has said "Mathematics is a way to settle in the mind a habit of reasoning." Here the results are developed through a process of reasoning. There are only a few premises on which we base our reasoning. The conclusions follow naturally from the given facts when logical reasoning is applied to the same. The reasoning in mathematics is of peculiar kind and possesses a number of characteristics such as simplicity, accuracy, certainty of results, originality, similarity to the reasoning of life, and verification. These characteristics have been discussed in detail in the relevant chapters of the book.

INTRODUCTIVE AND DEDUCTIVE REASONING

Reasoning in mathematics is of two types, viz., (i) Inductive reasoning; and (ii) Deductive reasoning. In the beginning mathematics arises out of practical applications and it is mostly inductive and intuitive. "Mathematics in the making is not a deductive science; it is an inductive, experimental science and guessing is the experimental tool of mathematics. Mathematicians, like all other scientists, formulate their theories from hunches, analogies and simple examples. They are pretty confident that what they are trying to prove is correct and in writing these, they use only the bull-dozer of logical deduction."

INDUCTIVE REASONING

When the statements or propositions are based on general observation and experience, reasoning is called inductive. When we can show that a particular property holds good in a sufficient number of cases, we conclude that it will also hold good in all similar cases. This type of reasoning is known as inductive reasoning.

DEDUCTIVE REASONING

Deductive reasoning is based on self-evident truths, postulates, axioms, etc. It proceeds from a premisis. Here the statements are the products of mind. This reasoning consists in comparing the statements and drawing a conclusion therefrom. Whitehead has correctly emphasised the place of deductive reasoning in mathematics by saying "Mathematics in its widest sense is the development of all types of deductive reasoning."

Essentials of a Deductive system: Deductive reasoning in mathematics requires a number of essentials which have been discussed as follows :

1. *Undefined terms* : In every branch of knowledge there are always certain terms which defy definitions. In their case the definition if any fails to do justice to the concept concerned. The following three examples or Euclidean definitions will uphold the above statements :

Point : A point is that which has no part.

Line : A line is length without breadth.

Straight line : A straight line is that which lies evenly with the points on itself.

These definitions are far from clear in themselves. Modern mathematics has solved this difficulty by treating certain terms as undefined. Point, line and surface have been recognised as undefined terms in geometry. In algebra the terms set, number and variable are undefined.

2. *Definitions* : After taking some undefined terms we can now frame the definitions of technical terms. This we can do with the help of undefined terms, not-technical language and other defined terms. For example in the case of a triangle—"If A, B and C are three non-collinear points, then triangle ABC is the union of the line segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$." Note that in this definition of triangle, we have used the terms—Collinear, point, union and line segment.

3. *Postulates* : As soon as we have collected a number of undefined and defined terms we are in a position to make all the statements which we want to prove in the subject. Each statement to be proved is known as a proposition. While proving a proposition we base our arguments on the previously proved statements. But going back in the chain of propositions we reach the earliest ones prior to which no previously proved statement is available. For proving such initial propositions we have to depend upon some self-evident truths which are accepted as such without proof. These self-evident truths are known as postulates. These are the first principles which can be taken for granted. They are assumed to be true without any necessity of explanation or proof.

Early Greeks tried to make distinction between postulates and axioms. The postulates were considered to be general truths common to all studies and the axioms as the truths relating to the special study at hand. Later on these concepts were modified. Postulates were considered as permissible constructions and all other initial assumptions were taken as axioms. However, in modern mathematics, no distinction is made between the two. The words postulates and axioms are used synonymously.

Some of the postulates are given below:

(i) A straight line may be drawn from any point to any other point.

(ii) A finite straight line may be produced to any length in that line.

(iii) A straight line has one and only one middle point.

(iv) Two straight lines cannot intersect at more than one point.

(v) All right angles are equal.
(vi) A triangle has interior and exterior angles.
(vii) There is one and only one straight line through a point parallel to a given line.
(viii) A circle can be drawn with any point as centre and with any length of radius.
(ix) Two circles intersect at two points.

Euclid regarded most of the other initial assumptions as axioms and called them common notion. These were accepted as true because of their conformity with common experience and sound judgment. The important notions are given below :

(i) Things equal to the same thing, are equal to one another.
(ii) The whole is equal to the sum of its parts and is greater than part.
(iii) If $a > b$, $b > c$, then $a > c$.
(iv) Magnitudes of figures which can be made to coincide with one another are equal.

MATHEMATICAL LANGUAGE AND SYMBOLISM

Another most important characteristic of mathematics which distinguishes it from many other subjects is its peculiar language and symbolism. Lindsay says, "Mathematics is the language of physical sciences and certainly no more marvellous language was ever created by the mind of man." Man has the ability to assign symbols for objects and ideas. Mathematical language and symbols cut short the lengthy statements and help the expression of ideas or things in the exact form. Mathematical language is free from verbosity and helps in to the point, clear and exact expression of facts. For example, instead of saying that the square of the sum of two terms is equal to the sum of the square of the first term, square of the second term and double the product of the terms, we can simply write $(a+b)^2 = a^2 + b^2 + 2\,ab$ in symbolic form.

Mathematical results in their symbolic form help in solving numerous complicated problems. Most of the later progress in mathematics depends heavily on the learner's ability to employ mathematical language and symbolism. It is reasonable to mention here that most of the results of scientific inventions and discoveries are stated through mathematical language and symbolism.

Addition, subtraction, multiplication, division and equality are indicated by well known symbols. Some of the other important symbols are :

$>$	for greater than
$<$	for less than

$\angle$	for angle
$\parallel$	for parallel lines
$\equiv$	for congruency
$\sqrt{}$	for square root
Σ	for summation.

In fact, it is not possible to prepare a comprehensive and complete list of symbols used in mathematics. The students must be made familiar with them so that they are in a position to understand mathematical processes and conclusions and mathematical literature. Many of them lose interest in the subject because of their inability to understand mathematical language and symbolism. They cram the statements and processes and try to solve problems mechanically. Rather they should be enabled to understand and appreciate precision, brevity, logic, sharpness and beauty of mathematical language.

PURE AND APPLIED MATHEMATICS

Pure Mathematics

Pure mathematics involves systematic and deductive reasoning. It treats only theories and principles without regard to their application to concrete things. It is developed on an abstract, self-contained basis without any regard to any possible kind of practical applications that may follow. It consists of all those assertions as that if such and such proposition is true of anything, such and such another proposition is true of that thing.

Applied Mathematics

Applied mathematics is the application of pure mathematics in the service of a given purpose. It has some direct or practical application to objects and happenings in the material world. It plays a great role in the development of various subjects. Every discovery in science owes much to applied mathematics. Principles of applied mathematics have been useful in the investigation of such phenomenon as heat, sound, light, optics, navigation and astronomy. Applied mathematics is a part of mathematics definitely related to or suggested by some tangible situations, though not always intended for practical use. It is the connecting link between pure mathematics on one side, physical, biological, social sciences and technology on the other. It acts and reacts not only on science technology but also on pure mathematics e.g., space dynamics, ballistics, fluid dynamics, elasticity theory, theory of relativity, mathematical biology, and mathematical economics.

RELATION BETWEEN PURE AND APPLIED MATHEMATICS

In the beginning, man studied mathematical structures as found in his environment. By and by these turned into abstractions. For instance, take the case of notation system. Firstly man started counting with the help of concrete objects, then he developed number words for counting and this further led to the invention of abstract numerals 1,2,3. . . Very often certain structures are discovered in the social and physical sciences. The pure mathematicians try to develop some parallel structures in mathematics. They do not stop here. Rather they go on to discover more and more. Many theories and structures of pure mathematics have wide applications, which were not known at the time of their invention. For example, the theory of complex numbers was developed from the point of view of pure mathematics but now it finds intensive application in electricity, radio, related fields of physics and engineering. The reverse of this is also true. This is very well illustrated by Synge as follows :

(i) A dive from the world of reality into the world of mathematics.

(ii) A swim in the world of mathematics.

(iii) A climb from the world of mathematics into the world of reality carrying the prediction in teeth.

EUCLIDEAN AND NON-EUCLIDEAN GEOMETRY

EUCLIDEAN GEOMETRY

It goes after the name of Euclid, the famous Greek Mathematician, and refers to the general type of geometry first logically organised by him. It is the type of geometry still studied universally in the high schools in the form of plane and solid geometry. Euclid introduced certain fundamental concepts as point, line, plane, angle, etc. He assumed certain self-evident truths which he called axioms and postulates. His geometry consists in proving geometrical truths with the help of axioms, postulates and previously proved truths. It employs a process of logical reasoning. Some of his definitions are given below :

1. *Point* : A point is that which has no part.
2. *Line* : A line is breadthless length.
3. *Straight line* : A straight line is a line which lies evenly with the points on itself.
4. *Surface* : A surface is that which has length and breadth only.
5. *Plane surface* : A plane surface is a surface which lies evenly with the straight line on itself.

6. *Angle* : A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Some of his postulates are given below :

1. A straight line can be drawn from any point to any point.
2. A finite straight line can be produced continuously in a straight line.
3. A circle may be described with any centre and any distance.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which are angles less than two right angles. (This is known as the Euclids postulate of parallels).
6. There is one and only, one straight line through a point parallel to a given straight line. (This follows from the above postulate).

SHORTCOMINGS OF EUCLIDEAN GEOMETRY

It suffers from the following defects :

(i) Many concepts requiring definition or proof are overlooked.

(ii) It makes use of certain assumptions which are not put as postulates. These assumptions are used in the proofs of certain theories.

(iii) Some of Euclid's ideas are crude, e.g., he assumed that all angles must be less than two right angles.

(iv) Some of the postulates put forward are not true.

(v) It does not take into consideration measurement on the spherical surface of the earth.

(vi) The practical applications of geometry are neglected.

(vii) It does not help the pupil in appreciating the basic characteristics of the perfect science of geometry.

NON-EUCLIDEAN GEOMETRY

The non-Euclidean geometry arose from the discussion, extending from the Greek period to the present day. The starting point was the Euclid's famous postulate of parallels which was proved false. Euclidean geometry came under the criticism from the renowned mathematicians like K.F. Guass, Lobachevslki and Reisman. Guass was the first to prove the Euclid's postulate of parallels false. Various other assumptions of Euclidean geometry were also criticised and disproved.

Non-Euclidean geometry takes into consideration the measurements on the spherical surface of the earth. This makes considerable difference in the concepts, terms and propositions of geometry. For example, a surveyor uses formulas of Euclidean geometry to measure the area of a plot or of a farm. Theoretically he should use spherical geometry since his measurements are made on the spherical surface of the earth. But for a small area, the surface of the earth is almost plane.

In the non-Euclidean geometry, every straight line has a finite length equal to the length of half a great circle on the sphere. There can be an infinite number of straight lines through two points. The sum of the angles of a triangle is greater than 180° and it approaches 180° when the area of the triangle approaches zero. Again in this geometry Pythagoras theorem is not true. It becomes true only when the area of the right angled triangle approaches zero. Unlike the postulate of parallels, in this geometry there is no line through a point parallel to a given line.

MODERN MATHEMATICS

It will be interesting to bring in some topics and concepts of modern mathematics. It is beyond the scope of this book to discuss all of them in detail. Only some of them are discussed here in brief.

SETS

Set theory is one of the important aspects of modern mathematics. The study of various sets with different structures is the corner-stone of advanced mathematics. There is no doubt that the idea of set is basic to all mathematics.

A set is a collection of objects, also called aggregate. We speak of a set of the members of a team, set of instruments in a geometry box, set of items in a tea-set is said to be defined when we know which objects form it i.e., we know definitely its elements. Some examples of defined sets are given below :

(i) *The set of integers 1, 2, 3, 4,* or $A=(1, 2, 3, 4,)$

(ii) Set of all positive integers or
$B=(1, 2, 3, 4,\ldots)$

(iii) Set of all vowels or

C=(a, e, i, o, u)

(iv) Set of prime numbers, or
D=(2, 3, 5, 7, 11...)

(v) Set of all rational numbers, etc.

EMPTY SET, NULL SET, OR VOID SET

A set containing no element is called a void or empty or null set and is denoted by the symbole; i.e.; Φ=( )

FINITE AND INFINITE SETS

A set containing only a finite number of terms is known as a finite set. One which is not finite is called infinite set. For example the set. (1, 2, 3, 4) is a finite set whereas the set of all natural numbers 1, 2, 3, 4. . .is an infinite set.

There are a large number of terms in this area which need to be known and understood. Some of them are : equal sets, equivalent sets, sub-set, universal set, power set, complement of a set, intersection of sets, union of sets, disjoint sets.

TOPOLOGY

Topology is derived from the Greek word which means a place. Previously it was defined as the study of a situation or position. It was known as position analysis in which the shape and size of configuration are unimportant.

It has its roots in the 9th century in the researches of Guass, Reismann and Cautor but it made great advances with the advent of the 20th century.

It is mainly concerned with the intrinsic properties of figures i.e., the properties of figures themselves and not the properties concerning their relationship to any surrounding space in which they may be embedded. Topology is qualitative mathematics which deals with the intrinsic qualitative properties of objects or figures. These properties are independent of size, location and shape. Intrinsic qualitative property is the property that does not change even when the object under consideration is subjected to stretching and bending without tearing. Without tearing is important because the points originally close together remain close throughout the process of stretching. The object whose intrinsic properties are to be studied may be anything—

a geometrical figure, rubber band, a collection of functions, an abstract space, etc.

Topology is sometimes known as rubber sheet geometry. If we draw certain figures on a rubber sheet and stretch it, the intrinsic property of the figure is unaltered. Technically speaking the figure is topologically unaltered.

The object of topology is to study the properties of figures i.e., connectivity, orientability, etc., that persist even when the figures are subjected to continuous deformation. If we imagine a circle made of pliable material, being subjected to continuous stretching and bending without being torn in the process, the circle is topologically unaltered. In topology, all such figures--circles, traingles, ellipses, polygons are topologically equivalent, because one figure can be transformed into the other without breaking or tearing. All such figures are known by the same term 'simple closed curve !'

In topology, the figures can be stretched, shrunk, bent, distorted or folded in any way, so long as nearly points remain close to one another. The points which are in contact remain in contact and the points not in contact cannot come into contact. In topology, neither breaks nor fusions can arise. Thus the transformation of a circle into a square or a triangle is a topological transformation, but the torus and sphere are topologically distinct. One cannot be transformed into the other without breaking or tearing.

Topology studies : (i) topological properties of figures; (ii) topological transformation; and (iii) arbitrary continuous transformation of geometrical figures.

In topology, we consider set of points. A point may stand for an object, a figure, a book, a dog, or a point in the sense of Euclidean geometry. Topology is the study of intrinsic qualitative aspects of points. The kinds of sets are known as spaces in topology.

ALGEBRAIC SYSTEMS

In order to understand algebraic systems, it is essential to have knowledge of sets, mapping and compositions. When a set is equipped with a binary composition or compositions, it is known as an algebraic system. Some of the algebraic systems are groups, rings, fields, vector spaces, etc. Let us take up these particular systems one by one :

Group : A set with composition is called a group, it satisfies the following conditions (adopting multiplication) :

(i) The composition is associative, i.e., (ab) c=a (ab) for every a, b, c belonging to G.

(ii) There is an identity e belonging to G such that ae=a=ea.

(iii) Every element of G has its inverse in G. That is for every a belonging to G, there corresponds an element b of G such that ab=e=ba.

The set of all integers is a group. This set consists of integers. . . —3, —2, —1, 0, 1, 2, 3...

Let us see if it satisfies the above conditions. First condition is obvious, because we know the relations of the form.

$(4+5)+6=4+(5+6)$

Here + is the composition as this corrects every ordered pair of numbers with another number of the same set e.g., $4+5=9$. Here +connects 4, 5 with 9.

As regards the second condition, see here 'O' is the identity.

$4+0=0+4$ or $5+0=0+5$

Also we know that $-9+9=0=9-9$

i.e., every element has the inverse. Hence the set of all integers forms a group.

The set of rational numbers also forms a group.

The set of all natural numbers is not a group because it does not satisfy the third condition.

Ring : A set of R with two structures or binary compositions (adopting additive) and multiplicative notation) is a ring if it satisfies the following conditions :

(i) The set R is a commutative group for additive composition :

Commutative property $a+b=b+a$ for every a, b, belonging to R.

(ii) Multiplication is associative i.e., $(ab)\,c=a(bc)$ for a, b, c belonging to R.

(iii) Multiplication is right as well as left distributive i.e., $a\,(b+c)=ab+ac$ (Right distribution) $(b+c)a=ba+ca$ (Left distribution).

All integers form a ring, because it is commutative group in respect to addition (Zero being the identity). Also the rational numbers and complex numbers form rings.

Field : A ring is called a field if it has at least two elements and

(i) is commutative

(ii) has unity (identity for multiplication)

(iii) has inverse for every non-zero element.

All rational numbers, all real numbers, all complex numbers form fields.

EXERCISE

1. What is the nature of mathematics ? Explain its fundamentals with the help of examples.
2. Discuss some of the important concepts of modern mathematics.
3. What do you understand by Euclidean and non-Euclidean geometry.
4. Describe briefly mathematical language and symbolism.
5. Write short notes on the following :

   (i) Inductive and Deductive reasoning.
   (ii) Postulates.
   (iii) Mathematical symbols.
   (iv) Pure and applied mathematics.
   (v) Euclidean geometry.
   (vi) Sets.
   (vii) Topology.
   (viii) Algebraic systems.

CHAPTER TWO

THE EDUCATIONAL VALUES OF MATHEMATICS

EVERY TEACHER OF mathematics needs to be informed and convinced about the educational value of his subject. Its educational soundness must be made clear to him at the very outset. His own conviction about its soundness enables him to convince the students, parents and the society likewise.

EDUCATIONAL VALUES

There are a number of questions which need to be answered at this stage. Why should everybody learn mathematics? Why should this subject be taught to everybody ? What is the place of mathematics in any scheme of education ? What is the importance of this subject in life and in school curriculum ? What shall be the advantage of devoting so much effort, time, and money to the teaching of mathematics ? What are the purposes and aims of teaching mathematics ? How does it make any contribution in the development of an individual ? It will be seen that these questions pertain to the same aspect viz., the educational values of mathematics.

A genuine teacher of mathematics will be interested in finding out the answers to these questions. He must feel convinced about the utility of his subject, so that he may be able to convince his students likewise The idea of the utility of this subject will develop in him keen interest in it. His students will copy his interest. He should be able to explain to the learner why he needs read this subject; that he will profit by its knowledge; and that ignorance of the subject will keep him handicapped. Something in the personality of the child will certainly remain unrealised in the absence of the study of Mathematics. The knowledge of its values and aims will stimulate and guide the teacher to adopt effective methods, devices and illustrative materials. These values etc., will lay down a criterion to enable the society and authorities to judge the success or failure of educational efforts. Adherence to these values and aims will make the teaching of Mathematics purposeful. Unless the goal is known, what hope can one have to find the path ? It is an admitted fact that a serious cause of deterioration in the present educational system is that its aims are not precisely defined.

VALUES AND AIMS

Knowledge of educational values helps the teacher to avoid aimlessness in teaching. Value is the springboard of aim and vice versa. There is nothing controversial about the two. One aims at a thing because one values it: or by aiming at a thing, one shall taste its value. "We aim at teaching Mathematics because we know its values or when we teach Mathematics in the light of its aims, we shall realise its values." Aimless teaching will realise no values.

What exactly is the responsibility of the schools towards the individual or what exactly would the schools do in order to increase the efficiency of the individual ? There are some definite motives for which the students are sent to schools. Schooling should be a purposeful affair. Everything done in the school should be the outcome of the aims and purposes of schooling. Whatever can contribute towards the attainment of these aims is included in the curriculum of the school. Educational value of any activity or subject can be judged from its suitability for the attainment of aims and purpose of schooling. Nothing done and taught in the school should pull in a different direction.

Broadly speaking there are three main considerations for which a child is sent to school. Education must contribute towards the acquirement of these values :

(i) Knowledge and skill.

(ii) Intellectual habits and power.

(iii) Desirable attitude and ideals.

These three values can be called utilitarian, disciplinary and cultural values of education respectively.

Provision of knowledge and skill is the practical or utilitarian value.

Provision of intellectual habits and power is the disciplinary value; and provision of training in desirable attitudes and ideals is the cultural value.

Is the study of Mathematics necessary for and helpful in the realisation of these values ? If Mathematics possesses these values, it is a most welcome part of education. In short here shall be discussed the existence or otherwise in Mathematics of the following three values :

1. Practical or utilitarian value.
2. Disciplinary value.
3. Cultural value.

1. PRACTICAL VALUE

One cannot do without the use of fundamental processes of this subject in daily life. A common man can get on sometimes very well without learning how to read and write, but he can never pull on without learning how to count and calculate. Any person ignorant of mathematics will be at the mercy of others and will be easily cheated. The knowledge of its fundamental processes and the skill to use them are the preliminary requirements of a human being these days.

A person may belong to the lowest or the highest class of society, but he utilises knowledge of mathematics in one form or another. Not to speak of an engineer, a businessman, an industrialist, a banker, a financier, a finance minister, a planner or a boss of any concern, even a labourer has got to calculate his wages, make purchases from the market, and adjust the expenditure to his income. If he is more sensible, he will believe in saving also and will lay by something for the rainy day. Whosoever earns and spends uses mathematics; and there cannot be anybody who lives without earning and spending.

Counting, notation, addition, subtraction, multiplication, division, weighing, measuring, selling, buying and many more are simple and fundamental processes of mathematics which have got an immense practical value in life. The knowledge and skill in these processes can be provided in an effective and systematic manner only by teaching mathematics in schools.

In many occupations such as accountancy, banking, shop-keeping, business, tailoring, carpentry, taxation, insurance, postal jobs, by which the needs of man are fulfilled, indirect or direct use of mathematics is made. These social agencies depends upon mathematics for their successful functioning.

It has become the basis of the world's entire business and commercial system.

It is very helpful in the study of various useful sciences.

Most individual and group projects in life fail for want of sense of calculation. A calculating person can anticipate all the possible handicaps to be faced and thus can adopt precautionary measures.

Ignorance of mathematics in the masses is a formidable obstacle in the way of a country's progress. Individual resources add up, to form national resources. The individual has not realised his responsibility towards building a sound national economy. Sometimes he is so miserably in debt, that he loses his entire property and even his means of livelihood. The family is left to starve continuously for many generations. Sometimes he is so well off temporarily that he spends blindly and wastes his hard earned money. He does not save. When he needs money, he falls a prey to exploiters. This is all due to his lack of sense of calculation and economy. He wastes his time, energy and money, without having any idea of the loss he is causing to himself and the nation at large. He is cheated and deceived in money matters; his ignorance of calculation often ruins him.

Budgeting is indispensable for the individual, the family, the society, and the State. There are the family budget, national budget, provincial budget, school budget, budget of a factory, budget of a business concern and so on.

Natural phenomena follow mathematical principles, e.g., the rising and setting of the sun and appearing of stars, the change of seasons, speed of rotation of planets, etc. Time is specified in each case.

One is sometimes misled about the practical value of mathematics on account of a feeling that whatever is taught in higher classes is of little use to a common man. A common man seldom uses that knowledge in his later life. But the value of the subject should not be measured only by the number of times one actually makes use of it, but by the number of times it can be made use of.

The real complaint is not that the subject is not of much use to an ordinary man, but that its present content and methods are not really useful.

The fault is to be attributed to the fact that Mathematics taught in the class-room is very often divorced from the mathematics of real life.

Even Napoleon said, "The progress and the improvement of mathematics are linked to the prosperity of the state". There can be no doubt about the validity of the statement that mathematical processes were invented in accordance with the requirements of mankind. Man cannot pull on without satisfying his needs. Thus mathematics will continue to occupy a prominent place in man's life. In all major and minor activities of life, such as making purchases in the bazar, arranging a party, admitting the child to a particular class, joining a profession, purchasing or selling a property, celebrating a marriage, mathematical considerations are uppermost in a human mind. In order to create system in life, we have to fix timings, prices, wages, rates, rations, fares, percentages, targets, exchanges, commissions, discounts, lengths, breadths, areas, volumes etc. In the absence of these fixations, life in the present complex society will revert back into confusion and chaos. The number imparts system to our life. In many types of our practical difficulties, number or measurements come to our help and solve the difficulties for us.

In this complex world passing through scientific and technological age, the practical value of mathematics is going to be increasingly felt and recognised.

2. DISCIPLINARY VALUE

According to Locke, "Mathematics is a way to settle in the mind a habit of reasoning." It trains or disciplines the mind. Due to its very nature, it possesses a real disciplinary value. It is exact, true and to the point knowledge, and therefore creates a discipline in the mind. Its truths are definite and exact. The learner has to argue the correctness or incorrectness of a statement. If taught in the right sense, it develops reasoning and thinking powers more and demands less from memory. The student comes to realize that thinking makes him a successful student of all the subjects. Its study results in the development of power rather than the acquisition of knowledge and knowledge also comes as a natural consequence or by-product.

Reasoning in mathematics possesses certain characteristics which are suitable for the training of the learner's mind. If properly emphasised and streamlined, these characteristics are likely to develop the corresponding habits in the learner. Here ensues a discussion of those characteristics and their influence.

(i) *Characteristic of Simplicity* : There is a vast scope for simple reasoning in this subject. It teaches that definite facts are always expressed in a simple language and definite facts are always easily understandable. So if you want to be understood, you must express yourself in a definite or simple way.

Moreover, one can easily follow a gradation going from simple to complex. The teacher advances by degrees to harder and harder portions. This procedure when practised for a pretty long time, becomes a habit.

(ii) *Characteristic of accuracy* : Without accuracy there is no chance of progress and credit in mathematics. Accurate reasoning, thinking and judgment are essential for its study. It is in the nature of this subject that it cannot be learnt through vagueness of thought and argument. In other subjects, it may sometimes be possible for the student to hide his ignorance by beating about the bush, but such tricks never pay in mathematics. Accuracy, exactness and precision compose the beauty of mathematics. The student learns the value and appreciation of accuracy and adopts it as a principle of life. He learns to influence and command others by his accuracy.

(iii) *Characteristic of certainty of results* : There is no place for subjectivity and proposal equation in mathematics. The answer is either right or wrong. Subjectivity of difference of opinion between the teacher and the taught. The student can verify his result by reverse process. It is possible for the child to remove his difficulties by self-effort and to be sure of the removal. The success of personal effort is a source of pleasure for him. He develops faith in self-effort which is the secret of success in life. He inculcates the habit of being certain about his achievement.

(iv) *Characteristic of originality* : Most work in mathematics demands original thinking. Reproduction and cramming of ideas of others is not very much appreciated. In other subjects, of course, ideas of others occupy a prominent place and have to be grasped by the student. Therefore he can safely depend on memory in other subjects; but without original thinking and intelligent reasoning there cannot be satisfactory progress in mathematics. When he has a new or a different mathematical problem, it is only his originality which keeps him going. The discovery or establishment of a new formula is also his original work. This practice in originality enables the child to face new problems and situations with confidence in his future career.

(v) *Characteristic of similarity to the reasoning of life* : Clear and exact thinking is as important in daily life as in mathematical study. Before starting with the solution of a problem, the student has to grasp the whole meaning. Similarly in daily life, while undertaking a task, one must have a firm grip on the situation. This habit of thinking will get transferred to the problems of daily life also.

(vi) *Characteristic of verification of results* : Results can be easily verified. As already pointed out, this gives a sense of achievement, confidence and pleasure. This verification of results is also likely to inculcate the habit of self-criticism and self evaluation. After making any attempt in life, the child would like to satisfy himself about its success or failure.

It may further be noted that the acquisition of knowledge is not the only purpose for which a boy is sent to school. Knowledge often becomes out of date, as these are the days of rapid progress and quick changes. In every field of life new ideas and new methods are being introduced rapidly. Therefore in this ever advancing society, the important thing is not only to learn facts, but also to know how to learn facts. The main thing is not the acquisition of knowledge but the acquirement of the power of acquiring knowledge. This is the discipline of mind, which everybody should aspire for.

Secondly, knowledge itself becomes real and useful only when the mind is able to apply it to new situations. In mathematics, there is again a vast scope for application. Ability to apply knowledge to new situations is inculcated in students. They acquire the power to think effectively. They learn to think and do not merely acquire mechanical habits. The intellectual power of the learner is strengthened. It generates the otherwise latent powers of thinking, reasoning, discovery and judgment of the individual.

3. CULTURAL VALUE

The understanding of the world in which man lives, of the civilisation to which he belongs and of the culture of which he is very proud, requires the understanding of scientific and social principles the development of which depends, in turn, upon mathematical principles. It has been truly said that "Mathematics is the mirror of civilization." Mathematics has got its cultural value, and this value is steadily increasing day by day. It helped man to overcome difficulties in the way of his progress. It has played a major role in bringing him to such an advanced stage of development. The prosperity of man and his cultural advancement have depended considerably upon the advancement of mathematics.

Modern civilisation owes its advancement to the progress of various occupations such as agriculture, engineering, surveying, medicine, industry, navigation, railroad building, etc. These occupations build up culture and they are its backbone. But one should not forget that mathematics contributes and has contributed extensively to the advancement of these occupations. Therefore mathematics shapes culture as a playback pioneer. Perhaps, the modern materialistic attitude in everything is the outcome of the deep influence of mathematics on life and culture.

Some of the important aspects of cultural heritage have been preserved in the form of mathematical knowledge only and learning of mathematics is the only medium to pass on this heritage to the coming generations. It is one of the repositories of precious and valued heritage.

Mathematics is also a pivot for cultural arts, such as music, poetry and painting. It might not be altogether a matter of chance that the Greeks, the greatest geometers, were also very successful in fine arts.

SOME MORE VALUES

In addition to these major and fundamental values, there are a few other values which are no less important.

DEVELOPMENT OF CONCENTRATION

Concentration is also a habit of mind and can be acquired through persistent practice. Every problem of life demands concentration. Mathematics develops it best. Mathematics cannot be learnt without whole-hearted concentration, in the absence of which the learner can't advance even a step further. A disturbed attention can take him nowhere. The learner, who can concentrate is likely to be a successful student throughout his career. A student who can concentrate well, can learn well.

According to Hamilton, the study of mathematics cures the vice of mental distraction and cultivates the habit of continuous attention.

ART OF ECONOMICAL LIVING

Living economically is a precondition for happy life. Economy in the matter of money is not enough, a person has to be economical in every single act of his life. Economy in the matter of time is most valuable. Indians have tendencies to gossip, to attend *melas* and celebrations for days on end, to sit daily for hours at a stretch in religious places, to go to meet friends and relatives and stay there for a number of days unnecessarily, to prolong matrimonial ceremonies and funeral rites, to waste time in joining processions and meetings which seldom follow a schedule. They seldom realise that time once past never returns. Similarly they seldom try to stick to time-limits in their actions and responsibilities on individual and national levels. There can be given innumerable instances of this wasteful and uneconomical attitude in many other things and activities. They don't understand that personal wastage is national wastage, and that personal economy is national economy. Even a speaker goes on dilating upon ideas and often fails to give anything concrete to his audience even by a long speech.

Mathematics is a very suitable subject for inculcating the spirit of economy. It deals with economy in time, speech and thought, and is bound to develop the corresponding attitudes in the individual. The learning of the art of economical living is a by-product of the learning of mathematics. The practices of thrift, sound investment and elimination of the superfluous, help in the development of this attitude.

POWER OF EXPRESSION

The habit of expressing oneself clearly and accurately is a great asset in practical life. In mathematics, one has always to be very careful while using appropriate words and terms. The habits of clearness, brevity, accuracy, precision and certainty in expression are formed and strengthened by the study of this subject. It is the constant exercise of these powers that produces the mathematical mind. One has got to use only such words as are clear and only such arguments as are certain. Its concepts and symbolism provide a means of concise expression which is elegant in its simplicity and exactness.

SELF RELIANCE

Learning by self-efforts is very important for a student of mathematics. The study of this subject is likely to develop the habit of self-education. By solving a problem correctly, the student develops self-confidence.

ATTITUDE OF DISCOVERY

Success in solving mathematical problems is not possible without the use of creative powers of the mind. Mathematical study trains the faculties of discovery and invention. The solving of a mathematical problem is accomplished on the same lines as the making of an invention.

UNDERSTANDING OF POPULAR LITERATURE

These days, the newspapers, journals, magazines and bulletins carry numerous mathematical symbols and terms in their matter. Reading of this popular literature is considered almost essential by every literate person. Even if one reads for the sake of pleasure, one cannot avoid coming across and grappling with some such symbols and terms.

QUALITY OF HARD WORK

Mathematics demands hard work from the learner. Hardworking citizens are very much needed in any developing society.

AIMS OF TEACHING MATHEMATICS

Since the elucidation of different aims will mean nothing more than the repetition of what has already been written under the caption 'the education values', the aims are listed here briefly :

1. To enable the child to solve mathematical problems of his daily life.
2. To develop in the child an acquaintance with his culture.
3. To provide a suitable type of discipline to the mind of the learner.
4. To prepare the child for technical professions such as those of accountants, auditors, bankers, surveyors, cashiers, engineers, scientists, statisticians and mathematics teachers.
5. To prepare the child for economic, purposeful, productive, creative and constructive living.
6. To develop in the child a sense of appreciation of cultural arts.
7. To prepare him for elementary as well as higher education in sciences, economics, engineering, psychology, etc.
8. To develop the habits of concentration, self-reliance and discovery.
9. To create in the child love for hardwork.
10. To develop in the child the powers of thinking and reasoning.

372.7
SID
S.C.E.R.T., West Bengal
Date 21-4-89
Acc. No. 4348
S.C.E.R.T., West Bengal
Library

11. To develop the learner's power of expression.
12. To enable him to understand and enjoy popular literature.
13. To develop in him a scientific and realistic attitude towards life.
14. To bring about all-round, harmonious development of the personality of the child.

AIMS OF TEACHING MATHEMATICS AT THE ELEMENTARY LEVEL

We have given above the aims of teaching mathematics in general. When we are required to enlist these aims at the elementary school level we have to think in a different way and our emphasis and priorities will vary. These aims can be enlisted as under:

1. To give a good start to the students in learning mathematics.
2. To provide to them clarity on fundamental concepts and processes of the subject.
3. To create in them an enduring interest and faith in the subject and to develop a love for it.
4. To introduce them to mathematical games, puzzles, recreations, hobbies and activities and to unravel before them the mysteries of the subject.
5. To develop in them a taste and confidence in mathematics.
6. To ensure in them accuracy and efficiency in fundamental processes and develop appreciation for accuracy.
7. To acquaint them with the relation of mathematics with their present as well as future life.
8. To develop in them the habits like regularity, practice, patience, self-reliance and hard work.
9. To prepare them for the learning of other subjects intimately related with mathematics.
10. To acquaint them with mathematical language and symbolism.
11. To prepare them for the learning of mathematics of higher classes.
12. To initiate and develop required discipline in the learner's mind.

THE PLACE OF MATHEMATICS IN EVERYDAY LIFE

Shut out mathematics from daily life and all civilisation comes to a standstill. In this world of today nobody can live without mathematics for a single day. Mathematics is intimately involved in every moment of everyone's life. Right from human existence on this earth, it has been a faithful companion. When man first wanted to answer the question : (How many ? How much ? How big ? How long ? etc.) he invented arithmetic. Algebra was devised to simplify arithmetical computations. For measurement and form geometry was invented. To find the position of high mountains and stars trigonometry was invented and developed and so on in the case of numerous other branches of mathematics. The knowledge of this subject was born out of felt needs of man. This knowledge is therefore indispensable. As the needs grow, the knowledge is bound to grow.

There is a definite need of mathematics in anybody's life-long planning and day-to-day planning. A human being is always after profitable and increasing returns. A mathematical approach is essential for any progress. Any approach devoid of mathematical considerations is likely to lead to decline. If anybody wants to make a success of his life, he must have recourse to mathematics.

Day-to-day evaluation in life or daily self-evaluation and over all life evaluation provide us assessment, judgment, guidance and direction for the future. This evaluation will have to be mathematical in nature to a great extent.

Even the most ordinary citizen has got to calculate his wages and buy things from the bazar. A person may be a mere housewife, farmer, labourer, shopkeeper, tailor, clerk, vendor, salesman, accountant, mason, driver, carpenter, or booking clerk, some knowledge of mathematics is absolutely necessary for him.

The entire atmosphere is surcharged with mathematics. The prices rates, discounts, commissions, rebates, interests, taxes, shortages, production, distribution, inflation, etc. are the issues with which everybody is intimately concerned. There is no escape from mathematical intricacies of life and livelihood.

All the illustrations and arguments in favour of practical value of mathematics can be utilised and repeated to justify the place of the subject in everyday life.

THE PLACE OF MATHEMATICS IN THE SCHOOL CURRICULUM

Everybody needs some knowledge of mathematics in one way or the other. But it is felt that for an ordinary man, the knowledge acquired during the primary and middle stage will suffice. Consequently there is a great controversy over making it optional or compulsory at the high/higher secondary school stage.

It is also believed that mathematics is an exceptionally difficult subject, i.e., its study requires special ability and intelligence; therefore everbody should not be burdened with the study of this tough subject; everybody is not able enough to learn it successfully.

References are made to low pass percentage in this subject in examinations.

Mathematics is a very useful subject for most vocations and higher specialised course of learning. But everybody who is studying it in the school is not going to be an accountant, engineer or statistician. But at such an early stage of education it is difficult to know who is going to be an engineer or a banker. Psychologists, however, have shown that it is possible at this stage to determine what a student can be in later life. Therefore the duty of the school is to give to the high school student a broad view of what he is capable of achieving in future. He should get a broader course to be able to choose a suitable line out of that. At the university stage, most of the physical and social sciences require the application of mathematics. To deprive the student of the knowledge of this subject at the high school stage means narrowing the choice of vocation for him. Ignorance of mathematics will be a great handicap in the progress of his studies in many other subjects. Its study will benefit him to a great extent. High/higher secondary school education will remain incomplete and incomprehensive if mathematics is excluded from it. No other subject can be a substitute for mathematics.

Educationists have begun to feel that education upto the middle standard is not sufficient for the citizen of today. The period of compulsory education needs extension from the middle to the high school standard. Naturally on the extension of the period of compulsory education from eight to ten or eleven years, mathematics will have to be retained as a compulsory subject for the longer duration of compulsory education. The question of making it optional at high/higher secondary stage will not arise at all.

While making it compulsory, the interests of both types of student's will have to be safeguarded. So the syllabus shall contain such subject-matter as is useful for those who are going to discontinue their education after the high school stage and are entering into other professions. It shall also contain such knowledge as elucidates mathematical principles and processes so that the students who will later join a university class, shall not be then handicapped at the sudden rise in standard.

There cannot of course be two opinions about the necessity of diversification and specialisation in the final year of the higher secondary school stage. Here mathematics has to be treated as an optional and specialised subject like other subjects of study.

One should not forget that during the age of education of 3 R's, mathematics was one of the three, rather two, subjects of study; its importance in the present age is no less. There can be no true schooling without mathematics.

EXERCISE

1. Mathematics is an indispensable part of education, not only because of its great disciplinary value, but also because of its many other educational values.
2. Discuss briefly the educational values of mathematics.
3. Illustrate with examples the importance of mathematics as a school subject.
4. Even if we view from purely utilitarian point of view, the study of mathematics is a must for every common man of the modern era.
5. What are the aims of teaching mathematics in schools.
6. There are a number of misunderstandings in the minds of some parents about the utility of mathematics as a school subject for their wards. How will you try to convince them.
7. Enlist and discuss the aims of teaching mathematics at the elementary level.

CHAPTER THREE

HISTORY OF MATHEMATICS

MATHEMATICS HOLDS THE mirror upto civilisation. It is no exaggeration to say that the history of mathematics is the history of civilisation. Mathematicians can take pride in the fact that their science, more than any other's is an exact science, and that hardly anything ever done in mathematics has proved to be useless. The geometry of the Greeks and the arithmetic of the Hindus are as useful and admirable as any research of today. Mathematics has been a progressive science.

VALUE OF HISTORY

History of mathematics has not so far been given any place in its curriculum, simply because no time can be made available for its study when already heavy courses have to be covered. Moreover, there has not been any serious realisation of the benefit which can possibly be had from this study. But once introduced, it is sure to become a source of interest and pleasure to the learners. Its values can be explained as follows :

1. Mathematics will be presented as a dynamic and progressive subject, full of human interest.
2. It will be instructive and interesting; it will not only remind us of what we have, but will also teach us how to increase our store of knowledge.
3. It warns the learner against hasty conclusion.
4. Many mathematical topics can be better introduced in the class by discussing their history.
5. It saves the student from attacking an unsolved problem by the same method which has led other mathematicians to failure.
6. It is important also as a valuable contribution to the history of civilisation. Mathematical and physical researches are a

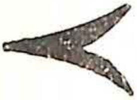
reliable record of intellectual progress. It is one of the large windows through which the philosophic eye looks into past ages and traces the line of intellectual development.

7. It reveals, that, at every stage, major or significant development of mathematics was conditioned by human needs.
8. Most of the terms, concepts and conventions can be properly understood only by reference to their historical background.
9. If the teacher uses his knowledge of the history of mathematics in the class-room, the students form a good impression of his study and knowledge. This helps him in commanding respect.
10. Gradation of the subject, correlation of mathematics with other subjects and psychological and logical order of the subject-matter—all these are done excellently.
11. It shows that mathematics is a man-made science. Its history will thus encourage the child also to contribute something to it.
12. It reveals that all the branches of mathematics were developed in relation to one another. So it guards the learner against compartmentalisation.
13. Some stories and events, mentioned occasionally, bring about a healthy change in the monotony of the class-room work.
14. It gives the impression, that mathematics has an intimate connection with other branches of knowledge, and should not be treated as an isolated subject.
15. It makes students appreciate the progress of man over the ages. They would like to read and hear how the old mathematicians discovered mathematical facts and tried their experiments.

THE ANCIENT CIVILISATIONS AND MATHEMATICS

Mathematics is a man-made science. Ancient men also felt their concern with this branch of knowledge. Though their conclusions were not very accurate, yet their efforts were quite serious and genuine. They were of course, motivated in their attempts by their social needs. Even in the most ancient times they must have done their best on the computational front of their life. They had their limitations, no doubt. Most of them made astonishing and interesting mathematical discoveries.

THE BABYLONIANS

The study of Babylonian mathematics begins with the notation of numbers. A vertical wedge stood for 1. while the charcacters the notation of numbers. A vertical wedge stood for 1. while the characters  and signified 10 and 100 respectively. In this connection the most surprising fact is that their notational system discloses the use not only of the above decimal system, but also of the sexagesimal one. There are two Babylonian tablets which exhibit the use of the latter. One of them, probably written between 2300 and 1600 B.C., contains a table of square numbers upto 60^2. The numbers 1, 4, 9, 16, 25, 36, 49 are given as squares of the first seven integers respectively. We have next $1.4=8^2$, $1.21=9^2$, $1.40=10^2$, $21=11^2$ etc. This remains unintelligible, unless we assume the sexagesimal scale, which makes $1.4=60+4$, $1.21=60+21$, $2.1=2\times60+1$. It can be noted here that the principle of position was also employed. But it is not certain that the Babylonians then knew the use of zero also. Their records of many centuries later of about 200 B.C.—give a symbol for zero which denoted the absence of a figure but apparently was not used in calculation.

The sexagesimal system was used also in fractions. Thus $\frac{1}{2}$ and $\frac{1}{3}$ are designated by 30 and 20 the reader being expected, in his mind, to supply the word "sixtieths". But what led to the invention of sexagesimal system ? It was assumed that the early Babylonians reckoned the year at 360 days, that this led to the division of the circle into 360 degrees and its further division into six segments of 60 degrees each.

It has also been found that Babylonians possessed the knowledge of multiplication and division tables, tables of squares and square root, a geometric progression, a few computations, and the rules for finding the areas of squares, triangles, and right triangles.

Most probably, Plato got the knowledge of the number from the Pythagoreans, and the Pythagoreans from the Babylonians.

It seems that in geometry they accomplished little. Like the Hebrews they took $\pi=3$.

Their astronomy has attracted much attention. They were the worshippers of heavenly bodies. They had some calculation about the new and full moon and the eclipses.

THE EGYPTIANS

All authorities concede that however far back one may go in Egyptian civilisation one finds no indication of an uncivilised state of society. The Egyptians built the pyramids at a very early period. Aristotle says that mathematics had its birth in Egypt, because there the priestly class had leisure needful for its study. Geometry is said to have originated in Egypt. The Egvptian King Sesostris divided the land among all Egyptians in order to give each one a quadrangle of equal size and to draw from each his revenues through a tax to be levied yearly.

A mathematical manual was written by Ahmes sometime before 1700 B.C. It mainly contains statements of results intended possibly to be explained by a teacher to his pupils. It deals with the area of an isosceles triangle, area of a circle ($\pi = 3.1604$, a very fair approximation), area of an isosceles trapezoid, the right triangle and conveys a rudimentary knowledge of proportion. The Egyptians were familiar, as early as 2000 years B.C., with well-known property of the right-angled triangle (for the special case at least, when the sides of a triangle are in the ratio 3:4:5). The Ahmes manual contains interesting information on fractions. Ahmes also gives the following examples involving; an arithmetical progression. "Divide 100 loaves among 5 persons; $\frac{1}{7}$ of what the first three get is what the last two get. What is the difference ?" He gives a correct solution of this problem. Again he speaks of a ladder consisting of the numbers 7, 49, 343, 2401, 16807. Adjacent to these powers of 7 are the pictures of animals and things, cat, mouse, barley, measure. Mortiz Cantor offers the following solution to this riddle : 7 persons have 7 cats, each cat eats 7 mice, each mouse eats 7 ears of barley, from each ear 7 measures of corn may grow. How many persons, cats, mice, ears of barley, and measures of corn, altogether? Ahmes gives 19607 as the sum of the geometric progression. Thus the manual discloses a knowledge of both arithmetical and geometrical progression. Ahmes also proceeds to the solution of equations of one unknown quantity. It thus appears that the beginnings of algebra are as ancient as those of geometry.

The period of Ahmes was a flowering time for Egyptian mathematics. It represents the most advanced attainments of the Egyptians in arithmetic and geometry. The principal limitation of their arithmetic was the lack of a simple, comprehensive symbolism. It is remarkable that they reached so great a proficiency in mathematics at so remote a period of human civilisation. But strange, indeed, is the fact that, during the next two thousand years, they made no progress in it at all.

THE GREEKS

About the seventh-century B.C., an active commercial intercourse sprang up between Greece and Egypt. There naturally took place an exchange of ideas as well as knowledge. Almost all the great Greek philosophers and mathematicians visited the land of the pyramids. To Egypt, Greece is indebted, among other things, for its elementary geometry. But this does not lessen the glory of the Greek mind. "Whatever we Greeks receive, we improve and perfect", says Plato. The Greeks felt a craving to discover the reasons for things.

THE IONIC SCHOOL

To Thales (640-546 B.C.) goes the honour of having introduced the study of geometry in Greece. He is said to have studied physical science and mathematics with the Egyptian priests; but he soon excelled his masters, and amazed King Amasis by measuring the heights of the pyramids from their shadows. It was done by using the knowledge of proportion. He is also known as the inventor of theorems on the equality of vertical angles, the equality of the angles at the base of an isosceles triangle, the bisection of a circle by any diameter, the congruence of two triangles having a side and the two adjacent angles equal respectively, the sum of the three angles of a triangle to be equal of two right angles and that all angles inscribed in a semicircle are right angles. With Thales begins also the study of scientific astronomy. He acquired great celebrity by making the prediction of a solar eclipse in 585 B.C.

THE SCHOOL OF PYTHAGOREANS

Having failed to found a school in his own country, Pythagoras (580? - 500? B.C.) quit his home and founded the famous Pythagorean school at Croton in South Italy. The school was like a brotherhood, its members were forbidden to divulge the discoveries and doctrines of their school to any stranger. This secrecy caused it to become an object of suspicion. People in Lower Italy revolted and destroyed his school. Pythagoras fled to Methapontum, where he was murdered.

At this school, mathematics was the principal study. Pythagoras raised mathematics to the rank of a science. His geometry is much concerned with areas (the famous Pythagoras theorem). He was especially fond of those geometrical relations which admitted of arithmetical expression. He was familiar with the construction of a polygon equal in area to a given polygon and similar to another given polygon. Pythagoras called the sphere the most beautiful of all solids, and the circle the most beautiful of all plane figures. The star-shaped pentagram was used as symbol of recognition by the Pythagoreans, and was called Health by them.

THE SOPHIST SCHOOL

In the year 480 B.C. the Greeks introduced democracy in their country, and therefore every citizen became a politician. Education had to be provided to every common man. Thus there arose a demand for teachers. The supply came principally from Sicily, where Pythagorean doctrines had spread. These teachers were called "Sophists or "Wisemen". They taught geometry, astronomy, philosophy and language.

The geometry of the circle, which had been entirely neglected by the Pythagoreans, was taken up by the Sophists. They made many attempts to solve the following three famous problems :—

(i) To "trisect an arc or an angle".
(ii) To "double the cube".
(iii) To "square the circle".

Hippocrates, a sophist, showed that circles are to each other as the squares of their diameters. Antiphon, Bryson, Zeno and Democritus were other famous mathematicians of this period.

THE PLATONIC SCHOOL

Plato was born at Athens in 429 B.C. and died in 348. He travelled widely and came in contact with great philosophers and mathematicians of other countries. He established his school in 389 B.C. and devoted the remainder of his life to teaching and writing.

He sought in arithmetic and geometry the key to the universe. He placed the inscription over his porch, "Let no one who is unacquainted with geometry enter here". He always stressed the continuous relationship between mathematics and philosophy. His school produced a large number of mathematicians. One of the greatest achievements of Plato and his school is the invention of analysis as a method of proof. Plato has been called a maker of mathematicians. Manaechmus, Dinostratus, Eudoxus, Leodames and many others were his students, who did research work in mathematics. Their researches pertained to prism, pyramid, cylinder, cone, parabola, ellipse, hyperbola, solid geometry, loci and astronomy.

THE FIRST ALEXANDRIAN SCHOOL

Ptolemey founded the University of Alexandria in about 338 B.C. It soon became a great centre of learning. Euclid was invited to open the mathematical school there. Ptolemy once asked him if geometry could not be mastered by an easier process, than by studying the elements. Euclid answered, "There is no royal road to geometry." It is a remarkable fact in the history of geometry that the Elements of Euclid, written over two thousand years ago, are still regarded by some as the best introduction to the mathematical sciences. Another prominent book of Euclid is the *Data* which is a course of practice in analysis. He wrote a few more books on : spherical geometry; astronomy; the hypothesis that light proceeds from the eye and not from the object seen; division of plane figures into parts in a given ratio. Some of his works on other subjects have been lost.

ARCHIMEDES

(287 B.C.—212 B.C.) was another famous and celebrated mathematician of those days. He wrote many books, namely (1) *Centres of Plane Gravities.* (2) *Quadrature of Parabol.* (3) *The Method*, (4) Two books on the *Sphere and Cylinder.* (5) *The Measurement of the Circle.* (6) *On Spirals.* (7) *Conoids and Spheroids.* (8) *The Sand Counter.* (9) Two books on *Floating Bodies*, (10) *Fifteen Lemmas.*

In the book *The Measurement of the Circle*, he proves first that the area of a circle is equal to that of a right angled triangle having the length of the circumference for its base, and the radius for its altitude. Aristotle knew the property of the lever, but could not establish its true mathematical theory. Archimedes' proof of the property of the lever holds its place in many text books to this day. His estimate of the efficiency of the lever is expressed in the saying attributed to him., "Give me a fulcrum on which to rest, and I will move the earth". His attention was first drawn to the subject of specific gravity when King Hieron asked him to test whether a crown, professed by the maker to be pure gold, was not alloyed with silver. His book *Floating Bodies* is a treatise on hydrostatics. He wrote on a very wide range of subjects, and is known as the Newton of antiquity.

APOLLONIUS

About forty years after Archimedes there flourished Apollonius who wrote eight books on 'Conic Section'. Besides the 'Conic Sections', Pappus ascribes to Apollonius the following works : *On Contacts, Plane Loci, Inclinations, Section of an Area, Determinate Section*—and gives lemmas from which attempts have been made to restore the lost originals.

The discoveries of Archimedes and Appollonius marked the most brilliant epoch of ancient geometry. Their separate fields of discovery may be designated by the names of geometry of measurements and Geometry of Forms and Situations or Geometry of Archimedes and Geometry of Apollonius.

Euclid, Archimedes and Apollonius took geometry to a high state of perfection.

About 100 B.C. there flourished Heron the Elder of Alexandria. But great uncertainty exists concerning his writings. It is not certain whether he was the real discoverer of the famous Heron's formula for finding the area of a triangle expressed in terms of its sides or whether the real discoverer was one Heron the Younger who might have lived later than Heron the Elder. Heron was a practical surveyor. Some of his formulas point to an old Egyptian origin. His writings satisfied a practical want, and for that reason were borrowed extensively by other peoples.

Hipparchus was the greatest astronomer of antiquity. He took his observations during the years 161 B.C. and 127 B.C. He is known as the originator of the science of trigonometry. He was mostly interested in mathematics as an aid to astronomical inquiry. He made arithmetical and also graphical devices for solving geometrical problems in a plane and on a sphere.

THE SECOND ALEXANDRIAN SCHOOL

This school may be said to have started with the Christian era. It was made famous by Claudius Ptolemy, Diophantus, Pappus, Theon of Smyrna, Theon of Alexandria and others.

The study of Platonism and Pythagorean mysticism led to the revival of the theory of numbers. This theory became the favourite study of this school.

CLAUDIUS PTOLEMY

A celebrated astronomer, was a native of Egypt. He flourished in Alexandria in 139 A.D. He wrote the famous work *Almagest* in 13 books. The fundamental idea of the Ptolemic system is that the earth is at the centre of universe and that the sun and planets revolve around the earth. He made maps of the earth's surface and of the celestial sphere.

PAPPUS

Probably born in about 340 A.D. in Alexandria, was the last great mathematician of the Alexandrian School. His genius was inferior to that of his predecessors. He wrote *A Commentary on the Almagest*, *A Commentary on Euclid's Elements*, and *A Commentary on the Aralemma of Diodorus*. His most well-known work is *Mathematical Collections*. It was originally in the form of eight books. It is worth noticing that he was the first to find the focus of the parabola. He also propounded the theory of the involution of points.

Theon of Alexandria brought out an edition of *Euclid's Elements* with notes, which he probably used as a text book in his classes.

DIOPHANTUS

He was one of the last and most fertile mathematicians of that age. We are constrained to remark that but for him algebra would have been almost an unknown science among the Greeks. It seems that he was the first Mathematician to perform such operations as $(x-1) \times (x-2)$ without reference to Geometry. The identities like $(a+b)^2=a^2+2ab+b^2$ were also proved by him algebraically. He dealt with the solution of simultaneous equations also.

Most geometers of the last 500 years showed lack of creative power. They were commentators rather than discoverers.

THE ROMANS

The sway of the Greeks was a flowering time for mathematics, but that of the Romans of sterility. As a consequence, not only the higher geometry of Archimedes and Apollonius, but even the *Elements of Euclid* was neglected. It is thought that even 'Roman Notation' was borrowed from some outside sources.

Romans had for long been concerned with payments of interest and problems on interest. The Roman laws of inheritance gave rise to numerous arithmetical problems. Here is an interesting and peculiar instance.

A dying man wills that if his wife being with child gives birth to a son, the son shall receive $\frac{2}{2}$ and she $\frac{1}{3}$ of his estates, but if a daughter is born, she receives $\frac{1}{3}$ and his wife $\frac{2}{3}$. It happens that twins are born, a boy and a girl. How shall the estates be divided so as to satisfy the will ? The celebrated Roman jurist, S. Julianus, decided that the estates shall be divided into seven equal parts of which the son receives four, the wife two the daughter one.

The only geometry known to them was a practical geometry which was employed in surveying. They had the formula for finding the area of a triangle and especially the approximate formula $\frac{1}{3}\,\frac{3}{0}\,a^2$ for the area of an equilateral triangle. Their king Julius Caesar ordered a survey of the whole empire in order to secure an equitable mode of taxation. With the services of Alexandrian astronomer Sosigenes, he also reformed the calendar In the later half of the fifth century, school books began to be compiled from the elements of Greek authors. These compilations are very deficient; but they are of absorbing interest, due to the fact that, down to the twelfth century, they were only sources of mathematical knowledge. Foremost among these compilers was Boethius. He wrote *Institute Arithmetica*, which is a translation of the arithmetic of Nicomachus, and a geometry based on Euclid's elements. A celebrated portion in his geometry is the one pertaining to an abacus where a considerable improvement on the old abacus has been made. Pebbles are discarded, and apices (probably small cones) are used. Upon each of these cones is drawn a numeral giving it some value below ten. These nine numerals are said to have been got by Alexandrians from the Hindus, in about the second century A.D. These were further passed on to the Romans on the one hand, and to the Western Arabs on the other.

THE CHINESE

The oldest Chinese work of mathematical interest is an anonymous publication, called *Chou-pei* and written before the second century A.D. This book was followed by the book *Arithmetic in Nine Sections*. This book begins with mensuration; it gives the area of a triangle as $\frac{1}{2}$ bh, of a trapezoid as $\frac{1}{2}(b+b')h$, of a circle variously as $\frac{1}{2}$ c, $\frac{1}{2}$d, $\frac{1}{4}$ cd, $\frac{3}{4}$ d^2 and $1\frac{1}{2}c^2$ where c is the circumference and d is the diameter. Then follow fractions, percentage and proportion, partnership, and square and cube root of numbers. Certain parts indicate a treatment of fractions. Division by a fraction is effected by inverting the fraction and then multiplying. There are given rules for finding the volumes of the prism, cylinder, pyramid etc. Then follow problems in allegation. There are indications of the use of positive and negative numbers.

Then comes the *Arithmetical Classic of Sun-Tsu* which belongs to the first century A.D. The author writes, "In making calculations we must first know positions of numbers. Unity is vertical and ten horizontal; the hundred stands while the thousand lies; and the thousand and the ten look equally and so also the ten thousand and the hundred." It is evidently a reference to abacal computation. Square root is explained more clearly in this book. Algebra is involved in this problem—"I don't know how many guests were there; but every two used a dish for rice between them; every three a dish for broth; every four a dish for meat; and there were 65 dishes in all."

Another book *Sea-island Arithmetical Classic* was written by Lin Hui in the third century A.D. He gives complicated problems indicating marked proficiency in abgebraic manipulation.

Of the books written during the succeeding centuries only a few are available. The book *Arithmetical Classic of Chang Ch' iuchien* of the sixth century can be mentioned. This contains an interesting problem—"A cock costs 5 pieces of money, a hen 3 pieces, and 3 chickens 1 piece. If then we buy with 100 pieces 10 of them, what will be their respective numbers."

In the first half of the seventh century, 'Wang H'siao-T'ung brought forth a work, in which the numerical cubic equations appear for the first time in Chinese Mathematics.

In 1247 Ch'in Chiu-shao wrote the *Nine Sections of Mathematics* which marked a decided advance on the solution of numerical equations. He made an advance by using 0 as a symbol for zero. It is probable that this symbol was an importation from India. Positive and negative numbers were distinguished by the use of red and black computing rods.

Half a century later, Chinese algebra reached its zenith in treatise *Introduction to Mathematical Studies* and *The Precious Mirror of the Four Elements* which came from the pen of Chu Shib-Chieh.

In the fourteenth century, astronomy and the calendar were studied. But after the well known achievements of the thirteenth century, Chinese mathematics for several centuries was in a period of decline. The "Systematised Treatise on Arithmetic," written in 1953, is famous for containing some magic squares and magic circles.

Christain missionaries entered China in the sixteenth century. They introduced European astronomy and mathematics, Euclid's geometry and logarithms.

THE JAPANESE

In about 552 A.D. Prince Sho'takm Taishi engaged his attention to Mathematics to such a degree that he came to be called the father of Japanese mathematics. A little later, Chinese science was imported. Bamboo computing rods were used in Japan in the seventh century. Numbers were represented by these rods in the manner practised by the Chinese. The Japanese 'soroban' was simply a more highly developed form of Chinese abacal instrument.

The Seventeenth century was a period of great mathematical activity. It started with Mo'ri Kambei Shigeyoshi who popularised the use of 'soroban'. His pupil Y. S. Ko' yu' wrote a book, *Jinko'-ki* in 1627, which is the oldest Japanese mathematical work now available. It deals with operations on the 'Soroban' including square and cube root. Another pupil Imamura Chi sho, published a treatise *Jugairoku* in 1639, which deals with the mensuration of the circle, sphere and cone. Isomura Kittoku, another author makes a crude approach to integration in his book *Ketsugisho* written in 1660. He gives magic squares, magic circles and also magic wheels. In 1666 'Sato' 'Seiko' wrote his *Kongedki* which, in common with other works of his day, considers the computation of π (=3.14).

We now come to Seki Ko'wa (1642-1708) whom the Japanese regard as the greatest mathematician of their country. The year of his birth was the year in which Galileo died and Newton was born. He was a great teacher. Like Pythagoras, he discouraged divulgence of mathematical discoveries made by himself and his school. He is said to have left hundreds of manuscripts; the transcripts of a few of them still remain. He published only one book, the *Hatsubi Samp ō*, 1674, in which he solved fifteen famous current problems. Noteworthy among his mathematical achievements are the 'Tenzan' method and the 'Yendan' method, which refer to improvements in algebra. Perhaps the most original and important work of Seki is the invention of determinants. Takebe was outstanding among his pupils.

The invention of the 'yenri' or 'Circle principle' is attributed to Takebe. This invention accomplishes almost the same things as are accomplished by the differential and integral calculus. Seki, Takebe and their co-workers dealt with infinite series, especially in the study of the circle and that of π.

In the early part of the nineteenth century there was greater infiltration of European mathematics than before. There was considerable activity but only one person of that century is worth mentioning. He was Wada Nei (1787-1840), who perfected the 'Yenri' still further, developing an integral calculus, and giving reasons where his predecessors gave only rules. He worked particularly on maxima and minima. After the middle of the nineteenth century the native mathematics yielded to a strong influx of Western mathematics. Undoubtedly Japan has produced some able mathematicians, but on account of her geographical isolation her scientific output did not affect or contribute to the progress of this science in the West.

THE HINDUS

After ancient Greeks, the first people, whose researches wielded a wide influence on the advance of mathematics belonged to the Aryan race living in far-off India. Hindus had scaled great heights in Mathematics and that was mainly to serve astronomy. The Greek mind was preeminently geometrical and the Indian mind arithmetical. The Hindus dealt with number, the Greeks with form. Numerical symbolism, the science of number and algebra attained in India far greater perfection than that previously reached in Greece. On the other hand Hindu geometry was merely mensuration, unaccompanied by demonstration. Hindu trigonometry is meritorious, but rests on arithmetic more than on geometry. There are evidences of some connection between Greek mathematics and Indian mathematics and of an intimate connection between Indian mathematics and Chinese mathematics.

Some of the remotely ancient mathematical works were composed sometime between 800 B.C. and 200 A.D. but most of them are now extinct. The works which merit notice and are now available, belong to the period from the 5th century onwards.

The noted Hindu astronomer Aryabhata was born in 476 A.D. at Pataliputra. His celebrity rests on a work entitled *Āryabhatiya*, of which the third chapter is devoted to mathematics. After him flourished Brahmagupta. In 628 he wrote his *Brahma-sphuta sidhanta*, of which the twelfth and eighteenth chapters pertain to mathematics. Probably to the ninth century belongs Mahavira, an author on elementary mathematics. He wrote the *Ganit-Sara-Sangraha* which throws light on Hindu geometry and arithmetic. In the following centuries Sridhara wrote *Ganit-Sara* and Padmanabha wrote on algebra.

It seems that the science made but little progress at that time; for a work entitled *Sidhanta Siromani* on astronomy, written by Bhaskara in 1150, stands little higher than that of Brahmagupta, written over 500 years earlier. The two most important mathematical chapters in this work are the Lilavati (the beautiful or the noble science), and Vij Ganita (the root extraction), devoted to arithmetic and algebra. It appears that the Hindus thenceforth contented themselves with the study of the masterpieces of their predecessors. Scientific intelligence went on decreasing.

The most ancient works *Sulvasutras* indicate that, perhaps as early as 800 B.C., the Hindu applied geometry in the construction of altars. The rules found in these works pertain to the construction of squares and rectangles, the relation of the diagonal to the sides, equivalent rectangles and squares and equivalent circles and squares. Some knowledge of the Pythagorean theorem is also disclosed in them. They yield a rectangle equal to a given square and a square equal to a given circle.

During the six centuries from the time of Āryabhata to that of Bhaskara, Hindu geometry dealt mainly with mensuration. But this geometry was without definitions, postulates, axioms and logical chain of reasoning. Āryabhata gave a rule for the area of a traingle which holds only for the isosceles triangle. Brahmagupta distinguishes between approximate and exact areas by giving Heron's formula $\sqrt{s(s-a)(s-b)(s-c)}$. Mahavira extended this formula and gave the area of an equilateral triangle as $a^2\sqrt{\frac{3}{4}}$. Brahmagupta and Mahavira made a further remarkable extension of Heron's formula by giving $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ as the area of a quadrilateral. Brahmagupta, however, recognised that this formula was true only for quadrilaterals that can be inscribed in a circle.

At one place Aryabhata gives an extremely accurate value of $\pi\left(3\frac{177}{1250} = 3.1416\right)$ but he himself never utilised it, nor did any other Hindu mathematician before the twelfth century.

The best achievement of the Hindus which has contributed most to the progress of intelligence is the perfection of the so-called "Arabic Notation". It appears that various numeral forms had been used long before the principle of local value and the zero came to be used. The principle of local value and the zero were probably known to Āryabhata. The symbol for zero was called 'sunya' (the void). It is to be found in the form of a dot in the Bakhshali arithmetic, the date of which is uncertain. The earliest undoubted indication of zero in India is in 876 A.D.

The ancient Hindus displayed great skill in calculating even with large numbers. Their modes of operation were different. They added the left hand columns first, and made the necessary correction as they proceeded. In subtraction, they had two methods. It was like this in multiplication—on multiplying with the next digit of the multiplier, the product was not placed in a new row, as is done today; but the first product obtained was corrected, by erasing whenever necessary, the old digits, and replacing them by new ones. The Bakhshali arithmetic contains problems such as: B gives twice as much as A, C three times as much as B, D four times as much as C; together they give 132; how much did A give ?" While considering some of the problems and methods of solving them, it is found that inversion was the favourite method with the Hindus. Āryabhata describes it thus, "Multiplication becomes division, division becomes multiplication, gain becomes loss and loss, gain." The Hindus solved problems in interest, discount, partnership, allegation, summation of arithmetical and geometric series, and devised rules for determining the numbers of combinations and permutations. It may here be added that chess, the profoundest of all games, had its origin in India.

The Indians were the first to recognise the existence of absolutely negative quantities. They brought out the difference between positive and negative quantities by attaching to the one the idea of "possession" and to the other the idea of "debts." The conception also of opposite direction on a line, as an interpretation of + and — quantities, was not foreign to them. They observed that quadratic has always two roots. Thus Bhaskara gives $x = 50$ and $x = -5$ for the roots of $x^2 - 45x = 250$. He further says "But the second value in this case is not to be taken for it is inadequate; people do not approve of negative roots". It shows that negative roots were known, but not admitted. The Hindus attempted problems involving irrational numbers also. For instance Bhaskara showed how, by the formula

$$\sqrt{a+\sqrt{b}} = \sqrt{a+\frac{\sqrt{a^2-b}}{2}} + \sqrt{a-\frac{\sqrt{a^2-b}}{2}}$$

the square root of the sum of rational and irrational numbers could be found.

"Indeed, if one understands by algebra the application of arithmetical operations to complex magnitudes of all sorts, whether rational or irrational numbers or space-magnitudes then the learned

Brahmins of Hindustan are the real inventors of algebra." In finding square and cube roots, they used the formula $(a+b)^3 = a^3+2ab+b^2$ and $(a+b)^3\ a^3+3a^2b+3ab^2+b^3$. They deserve credit for improvements in the solutions of linear and quadratic equations. Bhaskara says, "the square of a positive, as also of a negative number, is positive; that the square root of a positive number is twofold, positive and negative. There is no square root of a negative number, for it is not a square."

The glory of having invented general methods in the treatment of indeterminate equations, belongs to the Indians. The celebrated Greek mathematician, Diophantus was content with a single solution; the Hindus endeavoured to find all possible solutions. Probably the indeterminate equations grew out of problems in astronomy.

The Hindus evinced greater interest in trigonometry than in geometry. Interesting passages, are to be found in Varaha Mihira's *Pancha Sidhantika* of the sixth century A.D., which gives $\pi = \sqrt{10}$ $\sin 30^0=\frac{1}{2}$ and $\sin 60^0=\sqrt{1-\frac{1}{4}}$. But instead of dividing the radius into 60 parts in the manner of Ptolemy, the Hindu astronomer divides it into 120 parts. Applying the formula $\sin^2 a+\cos^2 a=r^2$, the obtained $\sin 45^0=\sqrt{\frac{r^2}{2}}=2431$, because the sine of 90^0 was equal to the radius or 3438. Substituting for $\cos a$ its equal $\sin (90-a)$, and making $a=60^0$, they obtained

$$\sin 60^0=\sqrt{\frac{3r^2}{2}}=2978.$$

Notwithstanding the Hindu indebtedness to other nations, the extent to which Indian Mathematics enters into the science of our times is remarkable. Both the form and the spirit of the arithmetic and algebra of modern times are essentially Indian. Unfortunately, some of the most brilliant results in indeterminate analysis, found in Hindu work, reached Europe too late to exert the influence they would have exerted, had they reached there two or three centuries earlier.

At the begining of the twentienth century, mathematical activity along modern lines sprang up in India. In the year 1907 there was founded the "Indian Mathematical Society", and in 1909 the *Journal of the Indian Mathematical Society* was started at Madras.

THE ARABS

"The Arabs were destined to be the custodians of the torch of Greek science, to keep it ablaze during the period of confusion and chaos, and afterwards to pass it on to the Europeans". This remark applies in part also to Hindu science. Thus science passed from Aryan (Hindu and Greek both) to Semitic races, and later came back to the Aryan.

The Abbasides at Baghdad encouraged the introduction of the sciences by inviting able specialists to their court, irrespective of nationality or religious belief. Medicine and astronomy were their favourite sciences. In the year 772 there came to the court of Caliph, a Hindu astronomer with astronomical tables which were ordered to be translated into Arabic. Certainly, the Hindu numerals, with zero and the principle of position also were introduced along with these astronomical tables among the Arabs. About the form and shape of the Arabic numerals, the statement of the Arabic writer Al-Biruni, who spent many years in India, is of interest. He says that the share of the numerals differed in different localities, and that the Arabs selected from the various forms the most acceptable.

But better information is now available about the way in which Greek science, dashed upon and penetrated Arab soil. In Syria the sciences, especially philosophy and medicine, were cultivated by Greeks. From there, Greek physicians and scholars were called to Baghdad. Caliph Al-Mamun (813-833) secured a large number of Greek manuscripts for translation into Arabic. At the beginning of the tenth century, the important Greek words could all be read in the Arabic tongue. The first Greek authors made to speak in Arabic were Euclid and Ptolemy. Among other important translations into Arabic were the works of Apollonius, Archimedes, Heron and Diophantus. It is thus evident that in the course of a century the Arabs gained access to the vast treasures of Greek Science.

In astronomy there was great activity in original research as early as the ninth century. The religious observances demanded by Mohammadenism presented to astronomers several practical problems. The prayers had to take place at definite hours during the day and night. The Mohammedan feast had to be held on exact days. For these reasons astronomical tables and instruments were perfected, observatories erected and a connected series of observations instituted.

Important to the student of Mathematics is Al-Khowarizmi's work on algebra and arithmetic. He lived during the reign of Caliph Al-Mamun. An Arabic writer says, "The arithmetic of Khowarizmi, being based on the principle of position and the Hindu method of calculation, excels all others in brevity and easiness, and exhibits the Hindu intellect and sagacity in the grandest inventions." His work the *Algebra* is very important. The title of this book is self-explanatory. It means "restoration and reduction". By "restoration" was meant the transposing of negative terms to the other side of the equation; by "reduction" the uniting of similar terms. It explains the elementary operations, and the solutions of linear and quadratic equations, and contains little that is original. It contains also a few meagre fragments on geometry. He gives the theorem of right angle triangle and calculates the areas of the triangle, parallelogram and circle . He prepared astronomical tables containing not only the sine function, but also the tangent function.

Next to be noticed is Tabit ibn Korra (836-901) who was born at Harran in Mesopotamia. He was proficient not only in astronomy and mathematics, but also in the Greek, Arabic and Syrian languges. His translations of Appollonius, Archimedes, Euclid, Ptolemy rank among the best. His dissertation on 'amicable numbers' (of which each is the sum of factors of the other) is the first known specimen of original work in mathematics on Arabic soil. He also trisected an angle. He is the earliest non-Chinese writer to discuss magic squares.

Foremost among the astronomers of the ninth century ranked Al-Battani. He was the first to prepare a table of cotangents. He dealt with horizontal and also vertical dials. Probably he knew the law of sines.

Abu'l Wefa (940-998) was another scholar of that region. He made the brilliant discovery of the variation of the moon. He translated Diophantus. He introduced also the secant and cosecant.

Al-Kuhi, another astronomer, was a student of Archimedes and Apollonius. He alongwith others made a study of the trisection of angles.

Creditable work in the theory of numbers and algebra was done by Al-Karkhi of Baghdad, who lived in the beginning of the eleventh century. His treatise on algebra is the greatest algebraic work of the Arabs. He was the first to operate with higher roots and to solve equations of the form $x^{2n}+ax^n=b$. He was also the first Arabic author to give and prove the theorems on the summation of the series :

$$1^2+2^2+3^2+\dots\dots\dots+n^2=(1+2+\dots\dots\dots+n)\ \frac{2n+1}{3}$$

$$1^3+2^3+3^3+\dots\dots\dots+n^3=(1+2+\dots\dots\dots+n)^2$$

The one who did the most to elevate to a method the solution of algebraic equations by intersecting conics, was the poet Omar Khayyam of Chorassan (about 1045-1123). He divides cubics into two classes, the trinomial and quadrinomial and each class into families and species. Each species is treated separately but according to a general plan. He rejected negative roots and often failed to discover all the positive ones.

Mathematics, which rose to great heights due to the contributions of Al-Karkhi and Omar Khayyam began to decline after them.

This much about the Arabs in the East should suffice. Between the Arabs of the East and the Arabs of the West, which were under separate governments, there generally existed considerable political animosity. There was less scientific intercourse among them than might be expected to exist between peoples having the same religion and written language. Thus the course of science in Spain was quite independent of the course of science in Persia. Going westward, one meets in Morocco a mathematician named Abu'l Hassan Ali, whose treatise "on astronomical instruments" discloses a thorough knowledge of the conics of Apollonius. When one reaches Cordova, the capital of Spain, one is struck by the magnificent splendour of her architecture. At this famous seat of learning, schools and libraries were founded during the tenth century.

Little is known of the progress of mathematics in Spain. The earliest name that has come down to us is Al-Majriti (who died in 1007), the author of a mystic paper on "amicable numbers". The only great astronomer among the Saracens in Spain was Jabir ibn Aflah of Sevilla, frequently called "Geber". He lived in the second half of the eleventh century. It was at one time believed that he was the inventor of algebra, and that the word algebra came from "Jabir" or "Geber". His chief work is on astronomy in nine books, of which the first is devoted to trigonometry. In his treatment of spherical trigonometry, he exercises great independence of thought. To the four fundamental formulae already given by Ptolemy, he added a fifth, discovered by himself. If a, b, c be the side, and A, B, C the angles of a spherical triangle, right angled at A, then $\text{Cos } B = \text{Cos } b \text{ Sin } C$. This is frequently called "Gebers' theorem.

It is a remarkable fact that, among the early Arabs, no trace whatever of the use of the abacus can be discovered. At the close of the thirteenth century, for the first time, do we find an Arabic writer, Ibu Albanna of Bugia (an African seaport) who uses processes which are a mixture of abacal and Hindu computation.

The latest prominent Spanish-Arabic scholar was Al-Kalsadi of Granada, who died in 1486. He wrote the book, *Raising of the Veil of the Science of Gubar*. The word "gubar" which originally meant "dust" stands here for written arithmetic with numerals. He devised symbols for the unknown, and used a considerable amount of algebraic symbolism.

The Arabs had the good fortune to possess rulers who, furthered scientific research. At the courts of the caliphes, scientists were supplied with libraries and observatories. They have to their credit several substantial accomplishments. They solved cubic equations by geometric construction, perfected trigonometry to a marked degree and made numerous smaller advances all along the line of mathematics, physics and astronomy. They adopted the learning of Greece and India, and kept what they received with care. When love for science began to grow in the West, they transmitted to the Europeans the valuable treasures of mathematics.

HISTORICAL REVIEWS OF DEVELOPMENTS

In the history of mathematics there have been some very remarkable developments in the form of discovery and evolution of certain ideas and processes. These ideas and processes claim special status and significance in the overall progress of mathematical knowledge. They are considered to be landmarks in the history of the subject. At this moment we shall discuss only a few of them.

(a) NOTATION SYSTEM

Number is one of the roots of mathematics, but it has taken roots and flowered into a full-fledged system over the period of centuries. The origin of notation system is as old as the man himself. The number sense is something innate in man. It is claimed that the birds and insects also have some number sense. Even the primitive man did not excel birds and insects in this number idea. This elementary concept took centuries to take any concrete shape.

The primitive man used various ways to count. He used fingers, notches, cuts in the trees, lines on the ground, pebbles etc. for the purpose. The notation system originated and developed differently in different countries.

BABYLONIANS

Represented numbers by wedge-shaped symbols. One was represented by V, ten was represented by < and hundred by V' <.

HEBREWS AND GREEKS

They used letters of the alphabet to denote numbers. For example X (alpha) stood for one, B (beta) stood for two and i (iota) stood for ten.

ROMAN SYSTEM

The Roman notation is based on the idea of counting by fingers or lines. Thus I, II, III represented one, two, and three respectively. V probably stands for the whole hand or the gap between the thumb and the four fingers. To avoid the clumsy I I I I for four, they wrote I before V. This symbol also gave rise to the idea of positional value. Similarly symbols VI, VII etc., were developed. The symbol X for ten seems to be the combination of two fives.

HINDU-ARABIC SYSTEM

Our present system in the form of numerals 1, 2, 3, etc., was originated by the Hindus. It was transmitted to the West through the Arabs and hence the name Hindu-Arabic system.

Some ancient symbols found carved on stones are :

I	II	+	6
One	Two	Four	Six

At some other places, some such symbols are found :

—	=	±	7	?
One	Two	Four	Seven	Nine

The early Hindus had no symbol for zero and they had different symbols for 10, 20, and 30 etc. Later on the symbol o was evolved by the Hindu mind, meaning 'Sunya', i.e., nothingness. One interpretation connects the concept of zero with the Hindu concept of Nirvana, the ideal state of nothingness. Arabs also made certain modifications in the Hindu numerals. During the 13th century the Hindu Arabic numerals became popular in the West and virtually spread all over the world.

By and by, certain modifications and improvements were effected and the Hindu-Arabic system assumed the present shape.

(b) WEIGHTS AND MEASURES

Before the evolution of the metric system of weights and measures, there was a wide variety of measuring units prevalent in different parts of the world. Due to lack of means of communication and exchange of experiences, different regions even in the same country used different units of measurement. In France, for example, there were eighteen different cloth measures and 400 ways of expressing the area.

For measurement of length, man used various limbs of the body e.g., foot, span, cubit, etc. yard came from the word yard which means a stick. In England a yard was fixed as the distance from the nose to the thumb of King Henry I. Ancient people in India also measured the yard similarly. The unit of weight 'Pound' was used by Romans. At some place, the pound was considered to be of 13 ounces, at another of 18 ounces.

So there was a lot of confusion in the system of weights and measures. In 1970, the king of France decided to have some uniformity in the system. He appointed a committee of scientists for the job and also invited other nations to send their scientists.

These scientists took the distance from the North pole to the equator on a line running through Paris and the one ten-millionth part of this distance was named as meter. However, it was found that the measurement of the meridian was not quite accurate. It was given up. Now the meter is defined as the length of a standard bar kept in Paris. For shorter measurements, the meter was divided by tens. Thus we have.

1 meter = 10 decimeters

1 decimeter = 10 centimeters and so on.

For longer measures, meter was multiplied by tens

10 meters = 1 decimeter

100 meters = 1 hectometer and so on.

AREA

Similarly units of area were developed on the same principle, and we have square meters, square decimeters and square centimeters as the units of area.

VOLUME

In the same way, units of volume were evolved which are cubic meter, cubic decimeter, cubic centimeter etc.

WEIGHT

The weight of one cubic centimeter of water at 4°C was taken as a gram. The gram was further sub-divided into decigrams, centigrams and milligrams and multiplied into decagrams, hectograms and kilograms.

SUPERIORITY OVER NON-METRIC SYSTEMS

The metric system of weights and measures is definitely superior to non-metric systems on various grounds. Firstly it has brought uniformity in the units of measurement all over the world. The varying non-metric units from place to place necessitated inter-system conversions and created confusion. It is easy to remember various metric measures as they are based on a scale of 10. The variety in nomenclature has also been simplified and has been made similar from measure to measure. The metric system facilitates the work of business and trade. The manufacturers and traders can deal and bargain on the basis of the same units throughout the world. The scale of ten facilitates calculations.

(c) LOGARITHMS

The word logarithm is the combination of Greek words 'logos' and 'arithmos'. Logos means to reason, to reckon and to calculate. Arithmos means a number. Thus logarithm stands for calculations involving numbers.

Logarithms were invented by a Scotsman, John Napier after many years of hard labour. The problems before him were to find out some way to cut short lengthy calculations. Napier spent about twenty years on the theory and finally logarithms were evolved. He got the idea of logarithms from the formula $\sin A \sin B = \frac{1}{2} [\cos (A-B) - \cos (A+B)]$. He, at first, evolved the logarithms of sines. The discovery of logarithms attracted the attention of the mathematicians all over the world. Kepler the famous astronomer who made use of logarithms to simplify his calculations said that Napier "tripled the life of an astronomer". Henry Briggs actually went to Napier and discussed with him the whole process. He explained the whole process of logarithms as under : Napier proceeded somewhat like this : $10^1=10$, $10^2=100$, $10^3=1000$ and so on

So Logarithm of 10 is 1
Logarithm of 100 is 2
Logarithm of 1000 is 3 and so on.

Further he observed that 100 × 100 means simply adding logarithm 2 and 1

$$10^2 \times 10^1 = 10^3$$

Similarly $10^2 + 10^1 = 10^1$

So division means the subtraction of logarithms.

Napier began to multiply, divide, square and extract square roots of various numbers. He computed logarithms of hundreds of numbers, and computed them to several decimal places. It took him months and years. Finally in 1614 the first table of logarithms of numbers was prepared.

Thus starting with the logarithms of sines, Napier applied logarithms to numbers. The terms Mantissa and Characteristic were introduced by Briggs in 1624. The first table of logarithms of trigonometric functions to the base 10 was made by Gunter in 1620. Gauss suggested using it for the fractional part of all decimals.

(d) COMPUTER MATHEMATICS

This is the age of automation. Man has invented machines for every field of life which make his functioning easy, quick and reliable computers are there to help a mathematician. They have their origin in man's attempts to devise ways and means to facilitate calculations.

The earliest attempt was probably a table of dust or sand on which calculations were carried out with the help of a stick. This was called abacus. Erasing on the abacus consisted in smoothing out dust or sand with hand. Next attempt was probably a ruled table with small sticks, pebbles or counters arranged in lines. Using the principle of position a single bead on a line would represent 1, in the next line it would represent 10 while an another line to the right, it would represent $\frac{1}{10}$ and so on. The ruled table was commonly in use in Europe until the beginning of 17th century and much later than that in countries like India, Mesopotamia, Egypt. A later form of abacus consists of a frame on which are stretched wires carrying movable beads or discs. With some experience, it is possible to add, subtract, multiply and divide more rapidly with this than with the help of pen and paper.

It will be interesting to note that about a hundred years ago Williams Shanks, an Englishman computed $\overline{\Pi}$ to 707 decimal places working with paper and pencil for 20 years. In 1950, a computing machine computed $\overline{\Pi}$ to 3000 decimal places in 13 minutes.

In 1617, Napier used a system of rods arranged to represent the method of calculations. The rods were known as Napier's bones. In 1620, Gunter made use of a number of lines for calculations. The length of the lines were proportionate to the logarithms of various numbers. By measuring various lengths on a pair of compasses, he was able to work out multiplications and divisions. This led to the invention of slide rule.

In 1642, Pascal at the age of 19 made a calculating machine to help his father in auditing government accounts. It was an adding machine adopted to the numbers of six figures. In the modern machines, carrying of ten is done mechanically. A disc engages a second disc, turning the latter one unit after 9 units are turned on the former.

The computer is developed from the calculating machines. It can perform a number of operations simultaneously whereas the calculating machine can perform only one operation at a time. One is simply to feed the computer properly and it gives the output i.e., solutions of problems correctly and in no time.

EXERCISES

1. Discuss the importance of history of mathematics for a teacher of mathematics.
2. How will you employ the history of mathematics for creating more interest for the subject among the students ?
3. Give a brief description of the contribution of Hindus towards the development of mathematics.
4. The subject of mathematics has attracted the attention of Indian scholars through the ages. Justify and illustrate.
5. Most of what we find in mathematics today is of Greek origin. Justify the statement.
6. Give an account of the progress made by mathematics under the important ancient civilisations.

7. Briefly trace the historical development of metric system of weights and measures.
8. Write short notes on the following :
    (i) Development of notation system.
    (ii) Development of logarithms.
    (iii) Development of computer mathematics.

CHAPTER FOUR

SOME GREAT MATHEMATICIANS

THE HEIGHTS WHICH mathematics is occupying today and the progress which it has made through the ages is all due to the dedicated and sustained work of its great students. A large number of them have immortalised themselves by their memorable contribution to their subject. Mankind will remember them forever with gratitude. Their lives and deeds will continue to inspire the students and scholars. These were the geniuses whose works will benefit the human race throughout its existence. Out of these shining stars of the world of mathematics we select the following few to get a glimpse of their greatness : (i) Aryabhata (ii) Brahmagupta (iii) Bhaskara (iv) S. Ramanuja (v) Euclid (vi) Pythagoras (vii) Gauss.

ARYABHATA

He was the first of the great Hindu mathematicians. He lived from 475 A.D. to 550 A.D. in or near the present city of Patna. He wrote on arithmetic including under this heading arithmetic and geometric progressions, quadratic equations, and indeterminate equations. He also computed arithmetical tables which contained a table of sines. His work in mensuration was crude and in this respect it somewhat resembled that of the Egyptians. For example, he gave the area of a triangle by a scheme that applies only to the case of an isosceles triangle.

He was also interested particularly in the series, permutations and linear and quadratic equations. He was familiar with the equations of the form $a x^2+b x+c=0$, but he did not tell how he approached their solution.

His work often called the *Aryabhatiya* consists of a collection of astronomical tables, and the *Aryastasata* which includes the *Ganita*, a note on arithmetic, the *Kalakriya* on time and its measure, and the *Gola*, on the sphere. The arithmetic carries numeration by tens as far as 10, treats of plane and solid numbers, and gives a rule for square root. His work is also noteworthy as containing one of the earliest attempts at a general solution of a linear indeterminate equation by the method of continued fractions. He tried his genius at the mensuration of solids also, but his work suffers from many inaccuracies. He gives the volume of a pyramid as half of the product of the base and the height, the volume of a sphere as $\pi\ 3/2\ _{r}\ 3$. He gives at one place an extremely accurate value of π viz. $3\ \frac{177}{1250}$ (=3.1416), but he himself never utilised it, nor did any other Hindu mathematician before the twelfth century. His rule about π is quite interesting—"Add 4 to 100, multiply by 8, add 62000, the result is the approximate value of the circle when the diameter is 20000. This makes π equivalent to $\frac{62832}{20000}$ or 3.1416.

It is Aryabhata who is learnt to have suggested the use of letters to represent unknown.

Aryabhata was one of those ancient scholars of India who could stand with pride among the greatest scholars even of the modern age. He was learned in philosophy and art and was a poet of excellence. He is hardly surpassed by any other in his treatise on mathematics and astronomy.

Aryabhata held that the earth was a spherical body and that it revolved around the sun. He had also made calculations of the rotation of the earth on its axis. He also appears to have had the knowledge of gravitational pull of the earth. His approach was scientific. He also deals with involution, area, circular arcs, algebraic equations, and identities.

BRAHMAGUPTA

He lived at Ujjain, the site of great astronomical observatory, in the seventh century. He gave formulae on area, which were faulty. He assumed, for example, that the area of an isosceles triangle was half the product of the base and one of the equal sides. He gave the rules for negative numbers in his algebra, and solve quadratics. In writing fractions, the scheme of writing the numerator above the denominator was used by him. He was particularly concerned with series and permutations. The rules of three in ratio and proportion was also known to him.

His knowledge of quadratics is shown in his solving the equation $x^2-10x=-9$ by a rule that is equivalent to the quadratic formula. He multiplied the constant term by the coefficient of x^2, added the square of half the coefficient of x and found the square root of this sum. He then subtracted half the coefficient of x and divided by the coefficient x^2. The quotient gave the solution of the equation.

He gave the rules of signs for their operation and distinguished between the two classes of numbers by writing a dot or a small circle over those that were negative.

He gave $\frac{22}{7}$ as the value of π and suggested 3 as a practical value

When he was only thirty years old, Brahmagupta wrote an astronomical work in twenty-one chapters entitled *Brahmasidhanta*. It begins by a definition of a calculator who is competent to study astronomy: "He who distinctly and severally knows addition and the rest of the logistics and the eight determinations, including measurement by shadow."

His arithmetic includes work with integers, fractions, progressions, barter, simple interest, the mensuration of plane figures, and problems on volumes and on shadow reckoning. He also states that the area of any quadrilateral whose sides are a, b, c, d, is,

$$\sqrt{(s-a)(s-b)(s-c)(s-d) \text{ where } s=\tfrac{1}{2}(a+b+c+d)}$$

a formula that is true only for cyclic quadrilaterals. However he has accurately given the formula for the triangular area as

$$\sqrt{(s-a)(s-b)(s-c)}$$

In the case of simultaneous equations of the first degree the unknowns are spoken of as colours, and the problems are chiefly astronomical. Indeed Brahmagupta was the first Indian writer, so far as we know, who applied algebra to astronomy to any great extent. One such problem is as follows:

"On the top of a certain hill live two ascetics. One of them being a wizard, travels through the air. Springing from the summit of the mountain he ascends to a certain elevation and proceeds by an oblique descent diagonally to a neighbouring town. The other, walking down the hill, goes by land to the same town. Their journeys are equal. I desire to know the distance of the town from the hill, and how high the wizard rose."

Brahmagupta was interested in indeterminate equations also.

The pleasing poetic garb in which all arithmetical problems are clothed is due to the Indian practice of writing all school books in verse, and especially to the fact that these problems, propounded as puzzles, were a favourite social amusement. Brahmagupta says in this regard—"These problems are proposed simply for pleasure; the wise man can invent a thousand others, or he can solve the problems of the others by the rules given here. As the sun eclipses the stars by his brilliancy, so the man of knowledge will eclipse the fame of others in assemblies of the people if he proposes algebraic problems, and still more if he solves them."

BHASKARA

He was known as Bhaskara the learned. He lived in Ujjain in the twelfth century. He was notable for his treatment of negative numbers, which he considered as debts or losses, and also for his treatise on arithmetic and measurement which he had named for his daughter Lilavati to console her for the accident which had prevented her marriage. Bhaskara, who was an astrologer as well as a mathematician, had discovered the propitious day and hour for this event. Any other time was prophesied as being sure to bring misfortune. While the bride was watching the hour cup whose sinking would mark the proper moment, a pearl dropped from her head dress and stopped the hole in the cup. The lucky moment passed unnoticed, and in compensation Bhaskara promised his daughter that he would give her name to a book which should last until the latest times.

Bhaskara was also concerned with series, permutations, linear and quadratic equations, mensuration and astronomy.

As an illustration of the properties of progressions, he gives the following problem:

In an expedition to seize his enemy's elephants, a king marched 2 Yojanas the first day. Say, intelligent calculator with what increasing rate of daily march did he proceed, since he reached his foe's city, a distance of 80 Yojanas, in a week ?

He gave $\frac{3927}{1250}$ (3.1255) as the value of π and suggested $\sqrt{10}$ for ordinary work.

He also gave a method for deriving sines for angles of every degree.

The special attention paid by him to the subject of permutations is evidenced by the following problems from the *Lilavati* :

"How many are the variations in the form of the god Sambu (Siva) by the exchange of his ten attributes held reciprocally in his several hands; namely, the rope, the elephant's hook, the dagger, the arrow, the bow as those of Hari by the exchange of the mace, the discus, the lotus, and the conch ?"

We may imagine that Bhaskara had in his mind different names of the god Hari and thought to astound his readers by the number of names that would have to be invented for Siva, had that god been given a name for each of the ways in which his ten attributes could be placed in his ten hands.

His most celebrated work *Lilavati* includes notation, the operations with integers and fractions, the Rule of Three, the most common commercial rules, interest, series, alligation, permutations, mensuration and a little algebra. The rules relating to zero are also given, to the effect that a+0=a, powers of 0 are 0, and a.0=0.

He also wrote *Bija Ganita*, a work on algebra. In this he discusses directed numbers, the negatives being designated as debt or loss and being indicated by a dot over each number, and the usual rules being stated correctly. The imaginary is dismissed with the statement, "There is no square root of a negative quantity for it is not a square." Where several unknown quantities are used, they are mentioned as colours. Surds are treated extensively. Besides numerous problems relating to geometric figures there are the usual poetic type.

A third work of importance written by Bhaskara is the *Sidhanta Siromani* (Head jewel of accuracy), in which he treats of astronomy and asserts the sphericity of the earth.

It is said that Bhaskara's grandson, Changadwa, was chief astrologer to King Simghana, and that in his time a college was founded to expound the doctrines of Bhaskara, which was a great tribute to his great memory.

The empirical nature of Hindu geometry is illustrated by Bhaskara's proof of the Pythagorean theorem. He draws right triangle four times in the square of the hypotenuse, so that in the middle there remains a square whose side equals the difference between the two sides of the right triangle. Arranging this square and the four triangles in a different way, they are seen, together, to make up the sum of the squares of the two sides.

Bhaskara gave a method of deducing new sets of solutions of ($x^2+1=y^2$ from one set found by trial). The problem is that of the equation : to solve ($x^2+1=y^2$ in integres x, y where C is a given non-square integer and $x, y, \neq, 0$). Bhaskara also discussed ($x^2+B=y^2$, C, B being non-square integers). His very elementary devices have excited the liveliest admiration. Bhaskara's attack on the equation is certainly the finest thing which was achieved in the theory of numbers before Lagrange.

S. RAMANUJAN

Ramanujan was a real genius. Though he lived only thirty-three years, but he has left a lasting and memorable impression in the world of mathematics.

He was born on December 22, 1887 in a village in Madras State. He was interested in mathematical studies even as a child and surprised his teachers with uncommon intelligence. His passion for mathematics and indifference towards other subjects of study resulted into his weakness in the latter. Since it was equally compulsory to pass in these subjects, he failed to get through even the intermediate examination. However this failure in the examination was in no way any index of his ability. But it cut short his educational career. In spite of that he devoted himself to the subject he loved with greater dedication and published several studies which were hailed in the mathematical world as unique.

He had a good deal of economic difficulty in the earlier stages, but in 1912, one Mr Walker came to his help. Mr Walker held an important post under the government. He was greatly impressed by

Ramanujan and obtained for him a scholarship of Rs 75/- per month from the Madras University. In 1914 he was invited to the Cambridge University to study mathematics at the expense of the University of Madras. He studied there upto 1917 and earned a name for himself in the whole of Europe. His articles were published in the best mathematical magazines of the world.

Ramanujan was one of the greatest pure mathematicians known to this world. His most outstanding contribution was in the field of theory of numbers. He belonged to that school of hyper-pure mathematicians who consider that mathematics like cricket is merly a game, which involves manipulation of symbols according to given rules. Dr Jagjit Singh writes—"Two corollaries follow from this tenet. First, like cricket, this game of manipulating symbols has no ostensible purpose except the fun of playing it. Secondly if by playing it for pleasure mathematics is kept sufficiently pure, it will at least be as clean. But it seems to me that in such an attempted rationalisation of what he (Hardy) and Ramanujan were doing, Hardy hurtled against the same difficulty as Tolstoy experienced when confronted with the ten-year old musician Fedka's question - "What is the use of singing?" It took Tolstoy 37 years to record the answer in his book—*What Is Art*. "For it required a deep probe into the mysterious relationship between art and life." He further writes "However, all mathematics, including hyper-pure branches which Ramanujan cultivated, is serious work and no serious work can be justified on such flimsy foundations."

Ramanujan's contribution is remarkable in another sense also. His game of pure mathematics secured for India prestige in the intellectual circles of England. He put India on the map of world mathematics in his own times, even as the unknown Hindu inventor of zero did many centuries ago. As a genius, devoted to the wonders of mathematics, he had desired that the mathematics he created should be allowed to remain clean, pure and unsullied by any contact with technological application. But it appears his fellow mathematicians and scientists are not prepared to grant him this privilege. Some of his theories are no doubt being used in the instruments of destruction which he would never have approved. However there are many instruments of peace also the preparation of which has been based on his mathematics.

He was preoccupied with mathematics right from his early formative college days. Too much of this preoccupation resulted in the neglect of English and other subjects. It had very tragic consequences for him. He failed in English and could not secure promotion to the senior class. As a result of this, the Subrahmanyam scholarship won earlier by him was discontinued. This scholarship was his only support for higher education. He was too poor to afford further education. Quite frustrated, he drifted into the Telugu country as a waif in desperate search for food for the preservation of his brain with which to cultivate mathematics. For many years he remained obscure and unknown and suffered from poverty and want. But when he was recognised as a great genius, it was too late. In the proper academic and scholarly atmosphere at Cambridge, he blossomed, but only for a while. His health had been damaged beyond repair. Moreover by the time he was honorably invited to Cambridge to study and work under Hardy, he had passed that early impressionable age in which along a born prodigy under proper guidance can amplify his natural powers and gifts.

Ramanujan was gifted with an astounding memory. He remembered the idiosyncrasies of every one of the first 10,000 integers to an extent that each one of them became his personal friend. Dr Jagjit Singh has beautifully summed up Ramanujan's play with numbers, thus: "Considering that a similar intimacy with 10,000 words would make one as great a master of language as Milton, it is no wonder that having diverted all his faculties to the field of numbers, he composed number rhapsodies which seem to those with an ear for the music of numbers to possess the same austere grandeur as the strident, sonorous notes of Milton's epic. One such lover of the music of mathematics was the great mathematician Hardy, who wrote—'I still say to myself when I am depressed and find myself forced to listen to pompous and tiresome people, well I have done a thing you could never have done and that is to have collaborated with both Littlewood and Ramanujan on something like equal terms." No greater tribute could be offered to the mathematical ingenuity of Ramanujan.

Goldbach's conjecture is one of the important illustrations of Ramanujan's contribution. The statement is that every even integer greater than two is the sum of two primes, that is numbers having no divisors. Thus 4 is the sum of the two primes 2 and 2, 6 of the primes 3 and 3, 8 of the primes 3 and 5, and so on. But any number of such illustrative examples whereby Goldbach made his guess do not prove that his statement is true of all the even numbers, although no one has yet found an instance to disprove it. A proof of this conjecture is still beyond the accumulated mathematical wisdom of the world. The nearest approach so far made is that of Vinogradow who, using methods traceable mainly to the work of Ramanujan and his collaborators, has been able to show that every large number can be written as the sum of at the most four primes (example $43 = 2+5+17+19$).

The partition of whole numbers is another similar problem which engaged the attention of Ramanujan. Take the case of number 3. There are three alternative ways to write it : $3+0$, $1+2$, $1+1+1$. You may easily verify that there are no other ways of partitioning this number, if we do not wish to use fractional numbers. Similarly 4 can be written in five different ways : 4+0, 3+1, 2+2, 2+1+1, 1+1+1+1, and there are no other ways if we decide to use only integers. It is easy to compute the number of partitions of any specific integer by a purely enumerative process like the one used above in calculating those of 3 and 4, the general problem is much more difficult. For, it requires the rule which may be valid for any number and universally applicable—no matter whether the number is 3 or 3,000,000. Ramanujan developed a formula for the partition of any number which can be made to yield the required result by a series of successive approximations. Although this problem of partitioning any number into its competent integers has proved to be very difficult, still pure mathematicians like Ramanujan have taken delight in further complicating the problem by enquiring into the number of ways of breaking any integer into the sum of two or more square, cubes etc.

When Ramanujan returned to India, Dr G.H. Hardy, the Head of the Department of Mathematics, Trinity College remarked "Ramanujan is returning to India as the greatest mathematician India has ever produced. I hope his country will give him a befitting reception." But alas the genius could not be spared to live for more than a few days and the cruel hand of death snatched him away from us too early.

EUCLID

Euclid (300 B.C.) has the distinction of being the only man to summarise all the mathematical knowledge of his times. Beyond the fact that he taught at Alexandria, we know nothing with certainty about Euclid himself. It is possible that he may have been an Egyptian and not a Greek. Who he was is of small importance, however, compared with what he accomplished in systematising Greek mathematics as it then existed.

It has been conjectured that he received his early training in mathematics at Athens and that there he came in contact with various attempts at reducing the subject-matter of geometry to a logical basis. At the invitation of the ruler Ptoelmy, he taught at Alexandria, and while there, he wrote several works, the most important being the *Elements*. It is said that Ptoelmy asked Euclid if there were not some shorter way to the study of geometry than through the thirteen books of the *Elements*, and that Euclid replied that there was no royal road to geometry. One of his students asked what he would gain from the study of geometry and in reply Euclid called to his servant, "Give this man three obols, since he must make gain out of what he learns."

Euclid's work was more formal than are the elementary geometries today. He was writing for mature thinkers who needed no introductory work to convince them of the value of the subject. Educators would say that his work lacked motivation—yet it has been the most widely used text that the world has ever produced. The concepts needed in each book were defined at the beginning. Postulates were stated and also those generally accepted hypothesis which we call axioms but which Euclid called "common notions". His *Elements* began with construction problems and the reader learned that a figure could be drawn with his instruments before he studied its properties. Each proof given in the *Elements* was stated in a standardised form : an enunciation of the problem in general terms, a statement of the conditions in specific terms (on hypothesis) the definition or specification of what is to be proved, the construction needed if any, the proof and the conclusion. At times, this would be followed by a statement of the condition under which the problem was impossible. These proofs were given systematically and there was no limit of the method by which they were first devised.

The extent of the subject-matter of the thirteen books of the *Elements* is given below:

Book I	Triangles, perpendiculars, parallels, areas of rectilinear figures, the Pythagorean theorem.
Book II	Transformation of areas, geometric algebra.
Book III	Circles, chords and tangents.
Book IV	Polygons and circles, construction of regular polygons including the regular pentagon.
Book V	Treatment of proportion.
Book VI	Idea of proportion applied to similar figures.
Book VII	Theory of numbers, classification of numbers as even, odd, odd times odd—plane, solid, perfect, etc. numerical theory of proportion.
Book VIII	Study of continued proportion.
Book IX	Number theory including a proof that the number of primes is infinite.
Book X	Study of irrationals.
Book XI	Solid geometry corresponding to Book I for plane geometry.
Book XII	The method of exhaustion used to show that circles are proportional to their diameters etc.
Book XIII	Regular solids.

The popularity of Euclid's *Elements* is evidenced by the fact that it was one of the first works to be translated from the Greeks to various languages. It was translated to the Arabic under the patronage of Harunal-Rashid. Its translation to Latin was among the first of the translations of the twelfth century since its first version in English. His book *Elements* is rightly called the greatest text book of all times. Over one thousand editions of his geometry have appeared in print since 1482, and manuscripts of this work have dominated the teaching of the subject for eighteen hundred years preceding that time. He is the only man to whom there ever came or ever can come again the glory of having successfully incorporated in his own writings all the essential parts of the accumulated mathematical knowledge of his time.

In addition to *Elements*, Euclid wrote a number of works. Among them are the Phoenomena, dealing with the celestial sphere and containing twenty-five geometric propositions ; the data, a course of practice in analysis, works on optics, and a work on division of figures, touching upon questions, which arise, for example, in surveying. There are some other works, namely, Catoptrica, containing propositions on reflections from mirrors; and Sectio Canonis, a work on musical intervals. Some of his lost works are Porisms, Fallacies, Comic Sections, and Loci on a Surface.

Some riddles are also known to have been formulated by him. One of them goes as follows : A mule and a donkey were walking along laden with corn. The mule says to the donkey, "If you gave me one measure, I should carry twice as much as you. If I gave you one, we should both carry equal burdens. Tell me their burdens, O most learned master of geometry."

PYTHAGORAS

The date and the place of his birth are both unknown. He seems to have been born between 580 and 568 B.C. Various stories are told of his parentage, but we are quite uncertain whether his father was an engraver of seals or a merchant. It is believed that he came from the Island of Samos near the mainland city of Miletus. At any rate he lived in Greece at the time of the dawn of the golden age. He lived in stiring times and was himself one of the great makers of the civilisation of his period. The spirit of the times was active in great works. Buddha was just promulgating his doctrine in India, and Confucius and Lao-Tze were laying the foundation for their philosophic cults in China and, whether or not Pythagoras came into personal touch with the Far East, he lived when the world was ripe for great movements.

He was a pupil of Thales. It is reported that after studying under his great teacher, he travelled to Egypt, and possibly he may even have come to India. He may have eventually settled in Crotona, a Greek city in Italy, where he gathered a group of wealthy men about him and founded the brotherhood of the Pythagoreans. This seems to have been a secret society with an siner group of men admitted to full membership and an outer circle of people who knew only part of the mysteries. His brotherhood has ever since served as a model for all the secret societies in Europe and America. Their badge was a five-pointed star and their studies were arithmetic, music, geometry, and astronomy. Their philosophy placed great importance on the idea of the infinite and taught the doctrine of transmigration of souls. Their astronomy had advanced to the point of considering that the earth was a sphere. Pythagoras based his philosophy upon the postulate that number is the cause of the various qualities of matter. This led him to exalt arithmetic, as distinguished from logistic, out of all proportion to its real importance. It also led him to dwell upon the mystic properties of numbers and to consider arithmetic as one of the four degrees of wisdom—arithmetic, music, geometry, and astronomy.

He asserted that unity is the essence of number, the origin of all things, the divine; that he had the idea of the limited and the unlimited; and that he held that from the latter came the ideas of time, space and motion. He attributed various numbers and forms to physical elements, for example, five is the cause of colour, six of cold, seven of health, and eight of love.

He is generally called the inventor of musical science or the harmonic canon. With his love for music and number it is natural to believe that he must have taken great pride to himself for connecting the two in the harmonic proportion. He seems to have held that the intervals between the heavenly bodies were determined by the laws of musical harmony, and hence arose the doctrine of the harmony of the spheres.

The difficulty of telling just what work was due to Pythagoras himself is increased by the fact that his follower tended to attribute their own discoveries to their master. It is evident, however, that Pythagoras had great interest in the properties of numbers, and that he devoted considerable attention to the study of areas, volumes, proportions, and the five regular solids.

He proved the sum of the angles of any triangle to be two right angles, and tradition has it that he sacrificed an ox on discovering the proof of Pythagoras theorem. His greatest service to mathematics was his inclusion of geometry as one of the liberal sciences essential to a man's education. The influence of the school of Pythagoras was so great that the group was dispersed for political reasons. Pythagoras died in exile, but his followers preserved his teaching and simplified his doctrines. Many great scholars of the times, like Plato and Zeno, studied under the Pythagoreans.

Pythagoras studied *perfect numbers* with veneration and interest. He travelled to Egypt with an urge to study under the Egyptian priests.

The Pythagoreans were the first to use geometry to express relations between qualities. The work would have been greatly simplified had they been in possession of algebraic symbolism, but even with the handicap of the clumsy expressions which this geometric algebra entailed, they were able to state important formulas and virtually to solve quadratic equations.

Under the influence of mysticism, the Pythagoreans believed the universe to be composed of five elements, one for each of the five regular solids. Thus the earth arose from the cube, fire came from the pyramid, air from the octahedron, water from the isosahedron, and the sphere of the universe from the dodecahedron.

By studying geometry for its own sake, the Pythagoreans, devoted considerable attention to matters of definition. They studied the properties of parallel lines, evolved a geometric algebra based on equivalent areas, worked with the five regular solids, and developed a theory of proportion which was discarded with the discovery of incommensurable quantities.

The Pythagoreans used symmetric arrays of dots to represent specific numbers, deriving the properties of various series of numbers from these diagrams. The addition of consecutive integers beginning with 1 made the triangular numbers 3, 6, 10 etc. The sum of consecutive odd integers gave the sqaure numbers 4, 9, 16 etc. The sum of consecutive even integers made the oblong numbers 6, 12, 20 etc. This system was extended to include polygonal numbers, which were really arithmetic progressions with the first term 1 and the common difference 3, 4, 5, etc. Solid numbers were made in a similar style and they may be visualised as symmetrical piles of marbles or of cannon balls.

Although he died away from his country in exile, but very soon the world realised the greatness in him. Two centuries later, the Government of Italy erected his statue in Rome to pay honour to "the wisest and bravest of the Greeks."

GAUSS

Gauss (1777-1855) was born at Brunswick. The marvellous aptitude for calculation of the young boy attracted the attention of Johann Martin Bartels, afterwards professor of mathematics at Dorpat, who brought him under the notice of Charles Williams, Duke of Brunswick. The Duke undertook to educate the boy, and sent him to the Collegium Carolinum. His progress in language there was quite equal to that in mathematics. In 1795 he went to Gottingen as yet undecided whether to pursue philology or mathematics. When nineteen years old, Gauss began jotting down in a copy-book very brief Latin memoranda of his mathematical discoveries. This diary was published in 1901. Of the 146 entries, the first is dated March 30, 1796, and refers to his discovery of a method of inscribing in a circle a regular polygon of seventeen sides. This discovery encouraged him to pursue methamatics. He worked quite independently of his teachers and, while a student at Gottingen, made several of his greatest discoveries. Higher arithmetic was his favourite study. In 1807 the Emperor of Russia offered Gauss a chair in the Academy at St. Petersburg, but by the advice of a friend, he declined the offer, and accepted a job at Gottingen. This was the post of the astronomer of the newly opened observatory. He spent his life here in the midst of continuous work. He had a strong will, and his character showed a curious mixture of self-conscious dignity and child-like simplicity. He was little communicative, and at times morose.

Although the work of Gauss belongs properly to the nineteenth century, he may well be considered as bridging the gap between work of the eighteenth century and the mathematical rigor of the nineteenth. Prior to his day, for example, scholars sought to prove Euclid's paralled postulate. Gauss, with his rare insight anticipated the discovery of a geometry that should not accept this postulate. He wroto in 1817, "I am becoming more and more convinced that the necessity of our geometry cannot be proved."

The son of a day labourer, he came from a poor family and owed his schooling to patrons who were impressed by his ability. During his years as a student at Gottingen, as mentioned earlier, he worked with the subject of least squares and showed how a regular polygon of seventeen sides could be inscribed in a circle by means of Euclidean instruments—the compasses and straight edge. His later researches included subject-matter from so many fields of mathematics that it is almost impossible to find a topic that is not closely connected with his work. He is rightly considered the real founder of modern German mathematics.

During his university career he conceived the idea of the theory of least squares, discovered the celebrated proposition that a circle can be divided into 17 equal arcs and began his great work on the theory of numbers. Thereafter he devoted his attention largely to the problems of electricity, and astronomy and also found time to write on the theory of surfaces, complex numbers, least squares, congruences, and hyperbolic geometry. He contributed substantially to every leading field of mathematics. He was one of the poineers to give serious thought to the question of a non-Euclidean geometry. He asserted that mathematics is the queen of the sciences, and the theory of numbers is the queen of mathematics." His first work on celestial mechanics made Laplace call him the leading mathematician of

Europe and in general he continued to enjoy this recognition from that time until his death. Kronecker said of him that "almost everything which the mathematics of our century has brought forth in the way of original scientific ideas, is connected with the name of Gauss," and the elder Bolyai spoke of him as "the mathematical giant who from his lofty heights embraces in one view the stars and the abysses."

A new epoch in the theory of numbers dates from the publication of his *Disquisitiones Arithmaticae*, *Leipzig*, 1801. The great law of quadratic reciprocity, given in the fourth section of this book, a law which involves the whole theory of quadratic residues, was discovered by him by induction before he was eighteen, and was proved by him one year later. The solution of the problem of the representation of numbers by binary quadratic forms is one of the great achievements of Gauss. He created a new algorithm by introducing the theory of congruences.

Gauss was led to astronomy by the discovery of the planet Ceres at Palerrmo in 1801. His determination of the elements of its orbit with sufficient accuracy to enable offers to rediscover it, made the name of Gauss generally known. In 1809 he published his book on astronomy, which contains a discussion of the problems arising in the determination of the movements of planets and comets from observation made on them under any circumstances.

In it are found four formulae in spherical trigonometry, now usually known as Gauss Analogies. He spent many years of hard work in the astronomical and magnetic observatory. He founded the German Magnetic Union with the object of securing continuous observations at fixed times. His researches on the theory of numbers were the starting-point for a school of writers.

Gauss is usually rated with Archimedes and Isaac Newton as one of the three great mathematicians in history. He was the first great mathematician to rebel successfully against intuition in analysis.

EXERCISES

1. Euclid's 'Elements' has been regarded as the most popular and universal text book adopted so far in the world of mathematics. Comment on this statement.
2. What do you know about Pythagoras and Pythagoreans ? With the distinctive contribution of both to the knowledge of mathematics ?
3. Discuss briefly the achievements of great Indian mathematicians. What is their place among the galaxy of the mathematicians of world fame ?

4. S. Ramanujan was the greatest Indian mathematician of this age, who revived with full glory the ancient Indian tradition of scholarship in mathematics. Discuss.

5. Write short notes on the following :

   (i) Gauss (ii) Bhaskara's *Lilavati* (iii) Aryabhata (iv) Brahmagupta.

CHAPTER FIVE

THE RELATION OF MATHEMATICS WITH OTHER SUBJECTS

MATHEMATICS AND PHYSICAL AND BIOLOGICAL SCIENCES

"ALL SCIENTIFIC EDUCATION which does not commence with mathematics is, of necessity, defective at its foundation."—Comte.

"A natural science is a science only in so far as it is mathematical."—Kant.

"Mathematics is the indispensable instrument of all physical research."—Berthelot.

"Mathematics is the gate and key of the sciences . . . Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of the world. And, what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy."—Roger Bacon.

"It is almost impossible to follow the later developments of physical or general chemistry without a working knowledge of higher mathematics."—Mellor, J.W.

"In mathematics we find the primitive source of rationality and to mathematics must the biologists resort for means to carry on their researches."—Comte, A.

For a long time, mathematics was taught because of the training in discipline it was supposed to give to the learner. But now, there has grown a different attitude towards it. This is the attitude of utility. As has already been pointed out, utility has not to be interpreted in a narrow sense. It has to cover the entire range of the possible uses of mathematics.

Mathematics has played a very important role in building up modern civilisation by perfecting all sciences. Even though, people have only a vague idea that all progress made by man is the result of scientific progress, they are strongly in favour of scientific and industrial education. This emphasis is confined to sciences such as physics, chemistry, biology, medicine and engineering. Mathematics, which is a science by any criterion and which rightfully belongs to this group, has not been accepted and emphasised as a science. It is an efficient and necessary tool which is employed by all these sciences and without which these sciences would not have made much progress. It has been very properly said about mathematics, "It is a science of all sciences and art of all arts." It is the pivot of all the sciences and arts. It may be a back-stage performer, but is a very powerful one.

For glimpses of its relationship with other sciences, for knowing its contribution to other sciences or for understanding the dependence of other sciences on it, here are given some examples from these subjects :

MATHEMATICS AND PHYSICS

Perhaps no other science is as close to mathematics as physics is. For higher education in physics, it is very helpful to possess a good qualification in mathematics. Only mathematical mind can take up the study of physics with confidence.

Pick up any standard book in physics, you will find that every rule and principle takes ultimately the mathematical form. Mathematics gives a final shape to the rules of physics. It presents them in a workable form. Mathematical calculations occur at every step in physics. Is is not mathematics that gives form and definiteness to the properties of matter ? Harnessing of nature is possible only through quantitative interpretations of ideas and imagination.

The units of measurement are employed to substances in physics as frequently as in mathematics.

The law of lever is based upon the simple mathematical principle of the balance of the sides of an equation.

Coefficients of linear expansion of different metals have been calculated, such as, coefficient of linear expansion of Iron has been found out to be .000012.

Similarly coefficients of cubical expansion of liquids have been calculated.

Charles's law of expansion of gases is based upon mathematical calculations. "All gases expand $\frac{1}{273}$ of their volume at 0°C for each 1°C rise in their temperature, provided they are kept at constant pressure."

Graduation of the stem of a thermometer and then the conversion of scales is also a mathematical work.

Tables of specific heat, Latent heat and Melting points have been prepared with the help of mathematics.

The Law of gravitation is given in the form of an equation :

$$F=G.\frac{m\,m^1}{r^2}$$

To find the resultant of two velocities, when a particle is under their influence, there have been deduced formulae, which only a good knowledge of mathematics can enable the learner to understand :

$$R=\sqrt{u^2+v^2+\angle uv\cos\theta} \qquad \tan\beta=\frac{v\sin\theta}{u+v\cos\theta}.$$

Further are given a few numerical problems to be done in physics, and these problems illustrate this relationship beyond any doubt :

Laws of Motion

A train of 120 tons mass running 45 miles an hour is pulled up in half a mile. What is the retarding force of the brakes in poundals and also in tons-weight.

Friction

Find the H.P. of an engine which is to move at the rate of 20 miles per hour up an inclined plane rising 1 foot in 100, the weight of the engine and the load being 60 tons and the resistance from friction 12 lb. weight per ton.

Liquid Pressure

A submarine weighs 2400 tons and has a volume of 112000 c. ft. Find the capacity of the blast tanks which must be filled with water so that the submarine may sink in water when desired. Take the weight of a cubic ft. of water as 62.5 lbs.

Expansion of Solids

A clock which keeps correct time at 25°C, has a pendulum of brass whose coefficient of linear expansion is .000019. How many seconds a day will it gain if the temperature falls to 0°C.

There can be given countless similar examples to prove beyond any doubt the dependence of physics on mathematics.

Take another instance viz., "Light changes its direction in going from a rarer to a denser medium." But unless it is known definitely how much it turns, what the relation is between the turning and the densities of the two media, the above property cannot be made use of in making eye-glasses or optical tsnemurtsni

Mathematics pervades physics so much that its impact and influence can be felt in every part of it.

MATHEMATICS AND CHEMISTRY

All chemical combinations and their equations are governed by certain mathematical laws. Formation of chemical compounds is governed by mathematical calculations. Water is a compound. Its formation is possible when exactly two atoms of hydrogen combine with one atom of oxygen. Without strict observance of this mathematical fact, the preparation is improbable.

In the manufacture of any chemical, there is some mathematical ratio in which different elements have to be mixed.

For estimation of elements in organic compounds, the use of percentage and ratio etc., has to be made.

Composition of air is another example. There are to be given the ratios of its different elements, when the composition by weight and by volume is to be interpreted mathematically.

Molecular weights of organic compounds are calculated mathematically.

The mathematical nature of the chemical equation can be further illustrated by the following examples :

$$2KClO_3+MnO_2=2KCl+MnO_2+3O_2$$
$$CaCO_3+2HCl=CaCl_2+H_2O+CO_2$$

MATHEMATICS AND BIOLOGY

Comte asserted, "The claim of any particular branch of natural philosophy to be considered as a science can be assessed only on the basis of the amount of mathematics employed in it." Knowledge of mathematics is considered essential for a biologist for two reasons. Firstly, biological study depends largely on its branches Biophysics and Biochemistry, which have attained a rank almost equal to that of independent sciences. It is necessary for the biologist to study mathematics for understanding these branches. It is indispensable for biophysics and biochemistry. Again, biomathematics is also growing as an important field of study for biologists.

Secondly, the biologist begins with his investigation with a series of direct observations of phenomena. The phenomena are described, classified and compared for generalising and deriving a biological law. At every stage of this process of investigation, mathematics is of immense help. Life processes are the most intricate of all natural phenomena. They have to be analysed and expressed clearly and concisely in a simple intelligible language. Mathematics provides symbols for a brief and precise expression of ideas. The expression and the calculation of the results of the Mendelian Laws are a clear example. In fact, the first paper that Mendel wrote on his discovery was called "The Mathematics of Peas."

The application of Mathematics in Biology can further be illustrated with an interesting example on the phenomenon of respiration in rabbit.

"During normal quiet breathing about one-eighth of the gaseous content of the lungs is changed at each inhalation and exhalation. During violent exertion this may rise to one-half, but there is always 50 per cent of the total lung capacity in which the air is unchanged except by diffusion. Thus, interposed between the changing air and the absorbtive surface is still air, which is the actual mediator between atmospheric air and lung epithelium. That the diffusion process is efficient is shown by comparing the proportions of the gases in inhaled and exhaled air. Roughly, in place of the oxygen absorbed, an equal volume of CO_2 is given out. This is the respiratory exchange and the ratio CO_2/O_2 is the respiratory quotient; in the case quoted it is 4.16/3.96."

	Oxygen	Nitrogen	Carbon Dioxide	Water Vapour
Inhaled	20.96	79.00	0.04	Variable
Exhaled	16.5	79.5	4.00	Saturated

Experimentation in Biology requires analysis and isolation of the particular character that is to be experimented upon. Biological phenomena are so complex, and the required analysis and isolation are so difficult that it is impossible to bring many of them under experimental control. The method of statistics is the only way of attacking such problems. Gestalt used this method to the measurement of inheritance of stature of children and the stature of their parents (the stature of the two parents being averaged in each case to give the "mid parent"). He found this coefficient to be $\frac{2}{3}$.

The Schultz-Borissoff Law of the Action of Enzymes such as pepsin and rennin is expressed by the formula

$$x - K\sqrt{F}.\ gt.$$

Where x=amount of substance transferred.

t=time of transformation.

f=concentration of enzyme.

g=initial concentration of substrate (e.g. albumen of milk) and k is a constant.

The growth in weight of infants upto nine months is given by the formula, $\log \frac{x}{341.5-x} = K(t-1.66)$, where x is the weight in ounces and t the no. of months.

Mathematical processes and calculations have been applied to advanced studies in heredity, nutrition, growth, maturation, fatigue and many other branches of biology and physiology.

The caloric and nutritive values of the food articles have been calculated.

Rates of respiration, transpiration and supply of water in connection with living bodies have been mathematically interpreted.

Ted Mcdonald has given very interesting mathematical facts about the human figure. The proportions of the human figure are strictly mathematical. A human being's beauty and attractive form depends on his/her proportionate build-up. The whole figure is six times the length of the foot. Whether the form be slender or plump, this rule holds good. Any deviation from it is a departure from the highest beauty of proportion bestowed on a human body. The Greeks made all their statues according to this rule. The hand, from the wrist to the middle finger is one tenth of the whole stature.

Similarly we have a scientific calculation about normal weight. The weight in pounds is double the height in inches.

MATHEMATICS AND ENGINEERING

The use of mathematics in engineering is very well known. It is considered to be the foundation of engineering. That is why a good mathematics qualification is necessary for admission to an engineering course. Engineering deals with surveying, levelling, designing, estimating construction, etc. In all these processes, application of mathemetics is very important. By the application of geometric principles to design and constructions the durability of things constructed can be increased. With its help, results can often be verified in engineering.

You may pick up any book on engineering, you will find that it si full of mathematical calculations, though the calculations are of a different nature and the formulae used are also of a different nature.

Here is an example to illustrate the point. A steel rod, 1 "diameter and 10 ft. long is subjected to an axial pull of 5½ tons. If E=13000 tons/in.^{2}, calculate the elongation of the bar. Here E is the Young's modulus of Elasticity.

Any number of such exercises can be quoted from engineering. These can be understood and solved only by a mathematical mind.

MATHEMATICS AND AGRICULTURE

Agriculture as a science is going to depend extensively on mathematics. Without a calculating attitude towards this important means of livelihood desirable targets in agriculture cannot be attained. The common agriculturist knows only how to work endlessly in this vocation, but he lacks the sense of comparing the return with the amount of work put in. It is not a paying profession for a large majority of the people, yet they cling to it. They invest power and money, but derive little substantial gain.

Of this science there are certain aspects which need direct application of mathematics, such as, measurement of land or area, average investment or expenditure, average return or income, production per unit area, cost of labour, time and work, seed rate, manure rate, etc.

Progress of a farm can be judged by drawing graphs of different items of production.

MATHEMATICS AND SOCIAL SCIENCES

The views of some eminent thinkers on the relationship between Mathematics and Social Sciences are given below :

W.F. White says, "The social sciences mathematically developed are to be the controlling factors in civilisation."

Lord Kelvin "All great scientific discoveries are but the rewards of patient, painstaking sifting of numerical data."

L. Karpinsky "Analytical and graphical treatment of statistics is employed by the economist, the philanthropist, the business expert, the actuary, and even the physician, with the most surprisingly valuable results, while symbolic language involving mathematical methods has become a part of well-nigh in every large business."

MATHEMATICS AND ECONOMICS

Mathematical language and methods are used frequently in describing economics phenomena.

A training in mathematics is of great help in acquiring command over exact language for expressing clearly some general relations and some short processes of economic reasoning.

Use of mathematics is now considered to be so essential for the study of economics that in most of the universities of the world a course in advanced mathematics is a necessary part of degree course in economics.

According to Marshall, "The direct application of mathematical reasoning to the discovery of economic truths has recently rendered great services in the hands of master mathematicians." Economics does not deal with the actual amount of money, production, capital, labour, utility, etc , but with how they behave in certain conditions and how they are connected with one another.

Statistical methods are applied to economic forecasts. Different issues of economics can be represented statistically, such as 'Trade Cycles', volume of trade, trend of exports and imports, population trends, industrial trends, thrift, expenditure of public moneys etc. The theory of probability is the basis of insurance.

Here is an example of direct application of mathematical reasoning. Demand for an article certainly depends on its price. It may also depend on other things. If D be the demand, P the price, it can be expressed as $D=f(P)$, provided other things are equal. PD is the total price of the article sold. It is not known what P is and what D is, but this relation holds good. From this relation the value of P can also be found out which will give the maximum profit to the producer. It will also be so when PD is maximum.

Professor Moore has arrived at certain results with regard to the prediction of the yield and the price of cotton. He has shown that his forecasts are exceedingly accurate. Professor Smith of the U.S. by following the same method of procedure, showed "there are certain critical periods in the growth of each crop, and he was able to locate these with much exactness. If to this is added a knowledge of the weather conditions, the importance of such discoveries for agriculture is obvious; a particular variety of crop and the time of its planting etc., can be then adjusted to suit particular places.

The businessman, who is the leader in the field of economic system, depends on statistical procedures to know about economic conditions and market trends. He deals with problems involving accounting, profit and loss, precentage, ratio, interest, taxes, etc. Knowledge of mathematics is necessary for understanding activities involving investments of money.

Popular literature and articles on economics make increasing demands on one's mathematical understanding and information. Common readers must have acquaintance with the common terms like average, ratio, percentage, variation, variable constant and formula. Such words daily occur almost on every page of the newspaper. Graphic and tabular interpretations of data depict economic conditions so simply and nicely that, without understanding them, a person cannot have clear comprehension of these conditions. Magazines, newspapers and bulletins are full of figures on accidents, governmental budgets and expenditures, employment, pro ducion, market, crops etc. They employ different numerical and statistical procedures. To keep one's knowledge up-to-date, one must study such literature and the mathematics applied in it.

MATHEMATICS AND PSYCHOLOGY

The great educationist Herbert has said "It is not only possible but necessary that mathematics be applied to psychology."

Now experimental educational psychology has become highly mathematical due to its concern with such factors as intelligence quotient, standard deviation, mean, median, mode, correlation coefficients and probable errors. Sooner or later, every science tends to become mathematical.

Arbuthnot says, "There are few things which we know, which are not capable of being reduced to a mathematical reasoning and when they cannot, it's a sign, our knowledge of them is very small and confused; and where a mathematical reasoning can be had, it is as great folly to make use of any other, as to grope for a thing in the dark, when you have a candle standing by you."

Statistical analysis is the only reliable method of attacking social and psychological phenomena. Until mathematicians entered into the field of psychology, it was nothing but a fight. Living organism cannot be easily isolated and experimented upon. Whenever any psychological experiment is made, it has to be made on large groups taken at random. No single individual can be representative of the whole species. For interpretation of a result, statistical analysis is necessary.

Suppose, you want to discover, whether ability is general or particular. General ability means, that if a student be brilliant in mathematics, he will be equally brilliant in language or history, or any other subject. Particular ability means, that if a student is brilliant in mathematics, it is not essential that he will be equally brilliant in language or history or any other subject.

The method of investigation is to be followed here. An adequately large number of pupils are taken. Carefully prepared tests in two subjects are given to them, say mathematics and language.

The pupils are arranged in their order of merit in both the subjects, separately. There can be four possibilities. Firstly, there may be perfect agreement or similarity between the two orders; i.e. the first boy in mathematics is the first boy in language and so on. In this case it can be said that the correlation coefficient between the ability in language and mathematics is plus one.

Secondly, there may be perfect dissimilarity. The first boy in mathematics may be the last in language and so on. Here the coefficient of correlation is said to be minus one. Here ability in language means want of ability in mathematics, and vice versa.

The third possibility is that there might be no correlation at all, i.e. the ability in one has no dependence on the ability in the other; a boy may have both, either the one or the other, or neither. In this case, the coefficient of correlation is said to be zero.

The fourth possibility and the most common type is what may be called partial or imperfect correlation. This will lie between zero and one, but unless it lies between 0.3 and one, after allowing for possible errors, one might as well say the coefficient is zero.

The finding of coefficient of correlation in this case is a mathematical problem. The science of statistics helps in solving this problem fairly accurately. Correlation coefficient is now widely used in the solution of psychological, economic and social problems. It is the most popular method used by Spearman and others for the investigation of man's abilities.

MATHEMATICS AND LOGIC

D'Alembert says, "Geometry is a practical logic, because in it, rules of reasoning are applied in the most simple and sensible manner."

Pascal says, "Logic has borrowed the rules of geometry. The method of avoiding error is sought by everyone. The logicians profess to lead the way, the geometers alone reach it, and aside from their science there is no true demonstration."

Geometry is a true demonstration of logic. Mathematics is the only branch of knowledge, in which logical reasoning or logical laws are applied and the results can be verified by the method of logical reasoning.

W.C.D. Whetham--"Mathematics is but the higher development of Symbolic Logic."

C.J. Keyser – "Symbolic Logic is Mathematics; Mathematics is Symbolic Logic."

The symbols and methods used in the investigations of the foundation of mathematics can be transferred to the study of logic. They help in the development and formulation of logical laws. In mathematics the symbol $<$ has got a meaning, e.g. $a<b$ means 'a' is less than 'b'. In logic, the meaning of this symbol has been extended. Let 'a' denote the class denoted by the cows and b, stand for the class denoted by the animals, then $a<b$, is easily interpreted to mean, "a is included in b", that is, all cows are animals.

For another example, take the symbol 'x'. Let A denote the class, 'Teachers' and B the class, 'Ladies.' AXB may be interpreted to mean the class of persons who are both Teachers and Ladies.

Thus the meanings of mathematical symbols have been extended to represent the relationship of propositions in logic.

The aims of the mathematician and those of the logician are practically the same.

MATHEMATICS AND PHILOSOPHY

The function of mathematics in the development of philosophical thought has been very aptly put by the great educationist Herbert, in his words, "The real finisher of our education is Philosophy, but it is the office of mathematics to ward off the dangers of philosophy."

Thomas Reid—"Mathematics have, in many cases, proved something to be possible and others to be impossible which, without demonstration, would not have been believed . . . Mathematics affords many instances of impossibilities in the nature of things, which no man would have believed if they had not been strictly demonstrated. Perhaps, if we were able to reason demonstratively in other subjects, to as great an extent as in mathematics, we might find many things to be impossible, which we conclude, without hesitation, to be possible."

A.N. Whitehead—"Philosophers, when they have possessed a thorough knowledge of mathematics, have been among those who have enriched the science with some of its best ideas."

J.S. Mill—"Mathematics will ever remain the most perfect type of Deductive Method in general; and the applications of mathematics to the simpler branches of physics, furnish the only school in which philosophers can effectively learn the most difficult and important portion of their art, the employment of the laws of the simpler phenomena for explaining and predicting those of the more complex."

A common man may ask, "How can there be any relationship between such widely different branches ?"

Mathematics deals with clear and definite things, whereas about philosophy it has been said in a jest. "When *A* talks to *B* about things which he does not understand and *B* replies wilh equal clearness, then they are said to be discussing philosophy." That is, they don't understand what they discuss, still they continue to discuss with all seriousness. They don't understand what they are talking and arguing, still they go on talking and arguing.

But mathematics occupies a central place between natural philosophy and mental philosophy. It was in their search of distinction between fact and fiction that Plato and other thinkers came under the influence of mathematics.

Usually, philosophy is defined as the science that investigates the ultimate reality of things. But the meanings of words, "ultimate" and "reality" are themselves controversial among philosophers. Each school of philosophy strongly maintains its own doctrines even though it is not able to convince the others. But the case is quite different in mathematics. This very difference in the nature of the subjects accounts for their mutual influence.

In mathematics, the philosophers find orderly and systematic achievements of unambiguous truths. By warding off the dangers. mathematics puts the philosophers on the right path of acquiring true knowledge. The philosopher finds that in mathematics one is able to rely on sheer thinking. By eliminating irrationality, mathematical methods produced the realistic school of thought in philosophy.

It has already been shown that mathematics has been very helpful in the development of various science at every deal by expelling superstitions from life, and clarifying various natural and social phenomena. The proper understanding, of the nature and composition of stars, of the nature of thunder and lightning, of theories about the atom, of diseases and their causes and treatment, has exercised a profound influence on the nature and scope of philosophy.

In fact, it is the mathematical development of the sciences that saved philosophy from degenerating into pure sensationalism and imagination.

Moreover, accumulation of ideas is not the aim of a philosopher. He wants to find the ultimate nature of the things, and for this he has to coordinate and synthesise different facts and bring all things into system. Thus he has to proceed according to mathematical procedures.

Philosophy owes a good deal to men like Pascal, Descartes and Leibnitz. They were all great mathematicians first.

MATHEMATICS AND FINE ARTS

Here are given a few prominent views on the relationship between Mathematics and Fine Arts :

Leibnitz, the great mathematician—"Music is a hidden exercise in arithmetic of a mind unconscious of dealing with numbers."

Boltzman — ". . . the simplicity, the indispensableness of each word, each letter, each little dash, that among all artists raises the mathematician nearest to the World-creator; it establishes a sublimity which is equalled in other art,—Something like it exists at most in symphonic music."

W.F. White—"Mathematics too has its triumphs of the creative imagination, its beautiful theorems, its proofs and processes whose perfection of forms has made them classic. He must be a "practical" man who can see no poetry in mathematics."

Pythagoras—"Where harmony is, there are numbers.

Bertrand Russell—"Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show."

Thoreau—"The most distinct and beautiful statement of any truth must take at last the mathematical form."

John Arbuthnot—"Truth is the same thing to the understanding as music to the ear and beauty to the eye."

Helmholtz—"The manipulations of artists' genius are but the unconscious expression of a mysteriously acting rationality."

Beauty of a piece of art depends on the manner in which it expresses truth. Mathematics is a knowledge of truth and realities. It is in itself a piece of fine art. It is a thing of beauty, and for many it is a joy forever especially when they do not study it for examination purposes. Something like mathematics exists in music. Everybody cannot appreciate fully a beautiful piece of architecture, a painting of musical notes. Only a mathematical mind can appreciate these arts with some sense of confidence. Mathematics provides a basis and background for aesthetic appreciation. Appreciation of rhythm, proportion, balance and symmetry postulates a mathematical mind.

EXERCISES

1. "Mathematics is a science of all sciences and an art of all arts." Comment on it and justify this statement.
2. Discuss and explain that mathematics has been a playback pioneer in the scientific advancement of the modern civilisation.
3. What is the relationship of mathematics with other subjects of the high/higher secondary school curriculum ? Give illustrations.
4. Mathematics has contributed immensely towards the enrichment and advancement of all the modern sciences ? What is your opinion about it ?
5. Even the social sciences of the modern era are becoming more and more mathematical. What is your acquaintance with this fact ?
6. Write short notes on :—
   (i) Relationship of Mathematics and Physics.
   (ii) Relationship of Mathematics and Economics.
   (iii) Relationship of Mathematics and Fine Arts.

CHAPTER SIX

CURRICULUM IN MATHEMATICS

"IT WOULD BE utterly contrary to the beautiful economy of nature, if one kind of culture were needed for the gaining of information and another kind were needed as a mental gymnastic."

—Herbert Spencer.

"Teach nothing merely because of its disciplinary value, but teach everything so as to get what disciplinary value it does have."

—Thorndike.

Any effort which is made intelligently will aim at certain results. If instruction in mathematics is to be worthwhile, it must be planned with the idea of achieving certain aims which represent values attainable from its study. It is thus possible to set up a rational criterion for determining suitable and useful content of mathematical courses. Selection and organisation of these courses and planning of student learning activities are extremely important though tedious undertakings. Selection of material should be controlled by purpose, which is to be served by mathematics in the general scheme of education.

The complexity of this task has further increased in recent years because of rapid social, technological and scientific changes. There have come about significant changes in man's outlook on life. The philosophy of life has changed and consequently the philosophy of education has also undergone a revolution. Curriculum has to be organised in the light of these changes. Tradition alone can no longer be regarded as adequate justification for its selection or organisation. The curriculum that suited a few years ago is not necessarily capable of satisfying our present needs. It must be able not only to identify the present needs of community but also to visualise the emerging needs in the modern commercial and the industrial field.

THE DYNAMIC APPROACH TO CURRICULUM

As already stated, tradition is not going to be our guiding principle in the task of selection and arrangement of the subject-matter

On the other hand, breach of tradition will be the rule. Revision of curriculum should be a continuous process. The existing subject-matter should be tried and its shortcomings noted for future modifications and improvements. We should never become complacent about it. Every year experts must confer and conclude about any modifications to be made in curriculum. It should not only be just workable but it should work well. Its difficulties and complications should be reduced, and scope widened after such annual stock-takings. It must be put to an annual check up. Any drastic or basic changes cannot of course, be made every year. Some of the loopholes can definitely be plugged and ground can be prepared for major changes.

THE TEACHER'S CONCERN IN CURRICULUM CONSTRUCTION

The teacher may genuinely feel that he is not concerned with this task. The curriculum is framed for him by others. Then, why should he bother himself with the study of the technique of curriculum construction ? His job is only to put into practice the content thought to be appropriate by experts. This theory has its roots in the past. The present day approach to curriculum construction is widely different. The teacher has to play a major role in this affair. His opinion has to be given maximum consideration at the time of revision or reconstruction of curriculum. He has not to remain indifferent and disgusted as he was in the past. He has to come forward with his suggestions so that curriculum may be made most effective and purposeful. Assimilation of the entire background and theory of curriculum construction is thus important for him. Only then he will be able to offer constructive suggestions for putting the curriculum on right lines. The new trend expects maximum partici-pation and contribution from the teacher's side. Therefore he must prepare himself well for this new responsibility. He should no more be an unhappy and helpless critic of the curriculum, but he should come foward with his own ideas and suggestions to solve practical difficulties.

OBJECTIVES OF CURRICULUM

A study of these objectives is obviously important at this stage. Without objections, selection and organisation may result in a planless, wasteful and ineffective course. The 'What' and the 'How' of mathematics have always been determined by its "Why". These objectives are fundamentally important as a guide to curriculum makers, text book writers, teachers and supervisory officers. If it is decided that mathematics as a subject is an essential part of school curriculum on account of its educational values, we have to adopt only those ways and means which help in realising those values. The educational values of the subject will guide us to make out the objectives of its curriculum. The aims of teaching mathematics have to be realised through the curriculum. It is the first practical step towards such realization.

• Young discusses the principal values of the study of mathematics under three general sub-heads (1) Practical values; (2) Mathematics as a mode of thought; and (3) Other functions. Under the third and rather indefinite heading he mentions values which are in the nature of attitudes, habits and ideals.

Breslich classifies the principal aims as (1) understandings, (2) skills, (3) problems and methods, (4) appreciations, (5) attitudes, and (6) habits.

Blackhurst lists them as (1) attitudes, (2) concepts, and (3) informations.

Schorling present them as (1) attitudes, (2: concepts, (3) abilities, and (4) information.

Mimick discusses them under four sub-heads as (1) practical values, (2) preparatory values, (3) cultural values, and (4) disciplinary values.

Since the objectives classified by Mimick are the most comprehensive, they can be adopted here for discussion. These values will then be the fundamental criteria to judge the suitability or otherwise of any topic. The topics will have to be selected or rejected accordingly.

CURRICULUM AS A MEANS TO AN END

Courses of study are rightly coming to be thought of as means to ends rather than as ends in themselves, and curriculum can be justified only in so far as it gives the promise of being the means through which certain desirable educational objectives can be attained. Curriculum is an instrument or a tool in the hands of the teacher to modify the child's behaviour. Adjust curriculum to suit the child rather than adjust the child to the curriculum.

FLEXIBILITY IN THE CURRICULUM

Curriculum should not be taken as a rigid pattern. It should rather be looked upon as a general outline of the material considered necessary as a part of the learner's training. There should be some freedom for the teacher and the taught to choose their subject-matter in the light of learning situations as they arise.

From the strictly practical point of view, flexible curriculum may not be a reality. In a class-teaching system, it is almost impossible to cater to individual aptitudes and differences. Either the selection and organisation of the content should be perfect or some freedom of selection and arrangement should be given to the teacher who, of course, would have to be a person of adequate initiative and resourcefulness.

Moreover, flexibility should not be taken to mean the absence of a system. Some minimum standards of attainment will have to be specified. There may not be any rigidity and enforcement in the matter, and the teacher may be told that he is free to make minor adjustments here and there. The entire selection and arrangement of the course content is a very heavy task and so cannot be assigned to an already overburdened teacher.

WHY REVISION OF THE PRESENT CURRICULUM IN MATHEMATICS ?

What is wrong with the existing Curriculum ? The curriculum has remained traditional in spite of ambitious and repeated efforts to improve it. The existing curriculum is far from being satisfactory. Prof. Kuppuswami has remarked very aptly that mathematics has been organised and taught in accordance with the ideas of the pure mathematician on the one hand and of the layman on the other. The mathematician has to be satisfied because, in many cases, it is he or those who are influenced by him that lay down the rules for the teacher. The layman has to be satisfied because ultimately it is he who pays the teacher. To satisfy the mathematician the content provides for the boys definitions, rules and reasoning processes. To satisfy the layman, the child is enabled to add, multiply and divide numbers and to commit to memory innumerable tables, rules and formulae. The layman may be told that his child will find use for all this in later life, that mathematics is a subject for the training of the mind and all that.

The teacher and the taught have been subordinated to some set processes of mathematics. The emphasis is only on the nature of the subject and future adult needs of the learner. The child receives a setback rather than education, in this process. The teacher remains indifferent and shows no enthusiasm about the curriculum.

Ordinarily, the official curriculum gives only the names of the topics to be done in different classes. The teacher cannot know what exactly he is expected to do. The present syllabus is anything but definite. The teacher is required to work out the details in his own way. It amounts to an excessive expectation from average teachers. Expert guidance is, therefore, needed; but it should come from a trained and experienced person rather than from one who has never taught the subject at school.

To form a clearer idea of detailed guidance to be provided by curriculum organisers, the teacher will please refer to later pages of this chapter.

WHO SHOULD ORGANISE AND REVISE THE CURRICULUM ?

Organisation and revision of curriculum should be done by a committee consisting of persons sincerely interested in the improvement of mathematical instruction. Big guns should be there, not simply because they hold high administrative positions, but because their interest in the subject is very practical and genuine. Field workers or teachers of mathematics should have a stronger voice in this committee. Some prominent businessmen, accounts officers, bankers and other experts may be taken on the committee. These members in particular and other teachers and administrators in general should be required to keep a regular record of their findings regarding any good points and drawbacks of the present curriculum. The committee should meet at least once a year to assess the situation and should then devote adequate time according to the volume of work. This work is very important and so should not be rushed through or done in a haphazard manner. Sometimes a change is proposed simply for its own sake. The work of curriculum revision needs proper handling by an expert and responsible committee.

The changes being brought about in the syllabi of other countries may also be kept in view.

PRINCIPLES GOVERNING CURRICULUM CONSTRUCTION

Mathematics is a very vast subject. The teacher does not have sufficient time to teach it as a whole. Hence the question of selection and choice of the course content. What should or should not be selected, is to be decided according to scientific criteria.

It has already been mentioned that there are four fundamental principles that should form the core of this criteria. The extent to which each of these should be depended upon, is now to be seen.

CRITERION OF PRACTICAL VALUE

"Knowledge for knowledge sake" is a noble ideal, but most students simply cannot afford to follow it. This ideal may be pursued in college education but not in school education.

Such knowledge as is applicable in daily life is everybody's necessity. Utility aims at intelligent, economical, well planned, well-balanced, civilised life. From the point of view of immediate utilitarian values, there are certain aspects of mathematics which are indispensable for the individual. These are of universal application and importance, and their acquisition by every child should be regarded as necessary. As illustrations of such distinctly practical value here may be mentioned the concepts of measurement : the four simple rules, percentage, average, profit and loss, interest, areas taxes, graphs, simpler notions of statistics, more common geometric figures, algebra of the formulae and the linear equation, mensuration, house-hold accounts, application of mathematical concepts and processes to problems of everyday life, and the mastery of mathematical vocabulary for intelligent reading of contemporary articles in current periodicals. These attainments should constitute a very-important part of the educational equipment of every citizen. The knowledge should not remain completely divorced from life. It should enable the pupil to solve problems that he actually meets in life.

The above mentioned topics directly represent practical values, and have been drawn almost entirely from the field of elementary mathematics. The word 'utility' should not be given a narrow interpretation. Giving a little broader meaning, it may be said that a topic should be included in the curriculum, if it is useful in everyday life, in the study of other subjects, in vocations, or if it is likely to help the pupil in appreciating the part played by mathematics in the development of civilisation in its various aspects—in commerce, trade, industry, engineering, physical and social sciences.

CRITERION OF PREPARATORY VALUE

The preparatory value is of two kinds (a) the part of mathematics in preparing the child for future life, and (b) the part of mathematics in preparing the child for future education. An interpretation of the first kind has been given in the section on the criterion of practical value. It is for the teacher to ensure that those who give up their studies after school stage, go fully prepared and equipped for future life. This group of students constitutes the majority and its needs should be given priority over those who would join the university.

Regarding the later part of this criterion, it can be said that a very small percentage of students go in for university education. Both the educationist and the administrator have come to realise that in secondary education there are other aims more important than the aim of preparing students for university education. Ways and means have to be devised to remove the dominance of college education over school education. It is not worthwhile to cater to needs of university education at the school stage, since most of the students are not going to benefit by that.

There is of course no denying the fact that something more than the minimum in mathematical instruction is not only an asset but a definite necessity at the collegiate and professional levels also. A student without the requisite background of high school mathematics, is often barred from entering courses which he likes to choose later.

Success in of sought courses in the college stage depends to a great extent on the degree of soundness of mathematical background made in the earlier stages.

In spite of these favourable arguments a topic having only preparatory value of this type is not entitled to occupy a place in the curriculum. The preparation for college education should be given during the college stage itself, because then it would be given to really needy and willing students. Nobody would feel compelled at learning such preparatory topics then. A topic may be helpful in the pupil's future education, but it should be useful in some other ways also.

CRITERION OF DISCIPLINARY VALUE

Any discussion of this value of mathematical instruction must take into account the question of transfer of training. Experiments by psychologists have demonstrated that transfer is never complete, automatic and inevitable. Many uncritical people have interpreted this to mean that transfer is nonexistent, although no experimental evidence has ever warranted this conclusion. Acceptance of the theory of transfer would mean that every mathematical topic should be given a place of high importance in the curriculum, regardless of any practical values that it might have, whereas denial of this principle would mean a consequent denial of any important place to topics other than those having distinctly practical or purely cultural values. The first viewpoint thus favours a broad comprehensive programme of mathematical instruction; the second advocates a definitely limited offering in this field.

Competent psychologists today are agreed that the truth about transfer lies between the two extreme positions of 'complete transfer' and 'no transfer'. It has been established that transfer takes place. The real question now pertains to the manner in which it takes place and to the conditions that are most favourable to it. The most reliable evidence indicates that, in order to achieve the disciplinary values of any subject, it is necessary to teach that subject with that specific end in view.

In particular, the ability to generalize meanings, symbols, relationships and processes and to apply such generalizations to new situations, represents transfer of the most genuine and vital sort. In fact, this is clearly what is meant by the expression "functional mathematics".

But it should not be forgotten that disciplining or training the mind is not the only function of education. By education useful and functional knowledge is to be imparted to the students. In the past this value was over emphasised, which consequently brought into the syllabus much useless material. For disciplining the mind one topic of mathematics is as good as another. So this quality of a topic cannot be made the sole criterion to justify for it a place in the curriculum. Unless and until there are some other virtues in a topic, mere disciplinary value does not or should not entitle it to a place in the curriculum. The topic must be useful to the learner in some other way also. There are so many useful mathematical facts to be learnt by the child during the limited time at his disposal, that it will be an irrational engagement for pupils to solve puzzles and riddles. When the same set of topics can provide both useful knowledge and mental training, it is a folly to make pupils learn one set on facts for mental exercise and another for the acquisition of useful information. Real and useful facts train the mind much better than unreal and useless problems. Unreal and fancy problems kill the critical attitude of the pupil.

Moreover, it is common knowledge that training of the mind largely depends on methods of teaching adopted in the classroom. If a topic is taught in an effective and logical manner, it will surely create certain mental habits. Therefore the disciplinary value of mathematics should be realised through suitable methods of teaching, and not through its acceptance as an important criterion for curriculum organisation.

CRITERION OF CULTURAL VALUE

Formation of desirable attitudes and ideals pertains to the cultural aim in education. But some people wrongly hold the view that less useful a piece of knowledge is, the more cultural it is. Classical, formal, unreal problems were considered to be cultural, whereas real life oriented topics were condemned as vulgar and narrow, for their stress on mere utility. The word 'cultural' was taken to mean ornamental. But in reality there is no such fundamental opposition between what is useful and what is cultural. For a piece of knowledge to claim a cultural status, it must clearly be useful. It must serve some purpose, otherwise it cannot find a place in culture. From this point of view, every piece of purposeful knowledge is of cultural value and vice versa.

The mathematical topics which can develop in a child the habits of tolerance, appreciation, sympathy, cooperation, accuracy and logical reasoning are no less important than those which claim to be useful in solving mathematical problems of daily life.

The subject of mathematics has played a great role in the advancement of human race and there may be many of its topics which have a history behind them. Cultural advancement owns a great deal to the discovery of certain mathematical facts. These facts are not going to lose their place, as long as society exists.

There may be certain ideas which were pursued by students of mathematics in ancient times but which were disproved later. The study of those ideas now will, no doubt, be of ornamental value only. But there are certain ideas which have become the life blood of modern culture and society. Their study, of course, will be a source of inspiration.

While making selection for the curriculum, the cultural value of a topic should not be ignored.

USE IS THE CHIEF CRITERION

The above discussion should lead the reader to conclude that "use" is the chief criterion. The modern man has become so utility-minded that he is not prepared to waste even a single moment in any non-utilitarian pursuit. Utility has become the driving force in his life. Moreover, the pre-specialization stage cannot be taken as the stage of preparation for specialization. The entire approach in curriculum construction has to be functional and purposeful. Useful information is a pre-requisite of useful life. Every attempt made and every minute spent has to be evaluated in the 'utility scale'. Useful knowledge has to be given top priority and, time permitting, other things may be considered in the end.

CHILD-CENTRED CURRICULUM

Though it has not been included in the list of fundamental crieteria, it is an equally fundamental principle for the selection and organisation of the course content. Today education is to be made child-centric. The days of subject centric education are gone. The child's present and future needs, his interests and aptitudes, his abilities and difficulties, have to dominate even the most insignificant aspect or activity of the educational process. The syllabus has not to be viewed from the point of view of the administrator, the mathematician and the teacher. Everything has to be made to suit the child. Every detail of the curriculum should be worked out in relation to the expected responses of the child. If the teacher can't make him like a particular portion, he has no right to thrust it on him. The curriculum is meant for the child's uplift and welfare, and it is subordinate to him. Its construction and revision should be governed by its workability with the child. It should be a child-centred scheme.

A COMPREHENSIVE CURRICULUM

A mere list of topics does not make a Curriculum. There should be necessary details also. It will be expecting too much from an ordinary teaching if he is required to work out the details for himself. When the same topic has to be distributed over a number of classes, the organisers should specify the points of 'start' and 'end' in each case. The teacher and the taught should know clearly what is expected of them.

Educationists and curriculum experts advise that teaching and learning processes of mathematics be made interesting and lively, but they never care to suggest how teaching of different topics can be made interesting. Too much is left for the teacher to devise and accomplish. Curriculum should provide a comprehensive picture of its contents. Techniques, life situations, material aids, related crafts or other activities and possibilities of correlation and other items that promote effective teaching should be listed in the curriculum. Curriculum should serve as a guide to teachers and authors of text-books. It should elucidate how the subject has to be made meaningful and comprehensive. As an illustration of this type of organisation, the following columns are suggested :

Name of the topic I	*Its sub-parts* II	*How to teach* III
How to learn IV	*Possibilities of correlation* V	*Material aids* VI
Activities or Projects VII	*Application* VIII	

With such an exhaustive plan of curriculum, the teacher and the taught will be obliged to adopt at least some of the proposed measures for effective teaching and learning. Students will also feel attracted towards, and interested in, the subject. The topics will appear real and meaningful to them. They will have something real and interesting to perform. Their learning will most likely be based on understanding and concrete experiences. Similarly the teacher will get expert guidance. His work will be facilitated a great deal. Something will be readily at his disposal to enable him to put life into his classwork.

PRINCIPLES AND METHODS OF ARRANGING THE CURRICULUM

After topics have been selected according to relevant fundamental principles, the work of arrangement of the topics starts. When and where a particular topic be introduced ? How should it be developed through different classes ? What should be done before and after it ? How much of it should be taught at a time ? All these and many more similar requirements have to be met through proper arrangement of the matter. Here follows a discussion of the principles and methods which are helpful in this case.

1. PSYCHOLOGICAL AND LOGICAL ARRANGEMENT

In the development of a particular branch of mathematics, the mathematician is chiefly concerned with a logical, rigorous treatment of the subject matter, whereas the teacher is usually concerned with its psychological organisation and presentation. It is the curriculum organiser who is called upon to integrate the two approaches. It may be remarked that the two do not differ substantially. The differences have been exaggerated and multiplied. The coalesce—the logical and the psychological methods merge.

There is no reason why the organisation cannot be both logical and psychological. The happy combination of the two is very desirable and feasible. All thinking is psychological. Education should make it logical as well. Psychology throws light on the use of a topic for the student from the academic as well as practical point of view. It takes into consideration the power of understanding and grasping of pupils in a particular age-level group. The order in which topics are to be taken will largely depend on its findings. Similarly logic must be there. Psychology should decide what kind of logic is appropriate for the pupils of a certain age and what type of topics will be most suitable for the development of such logical thinking. Logic will help in maintaining the link and sequence of topics found useful and meaningful for the child. In combination, these two view points can make the subject-matter interesting and comprehensible.

2. SCOPE FOR PRACTICAL WORK

All healthy and normal children like to be active. They derive pleasure from all activity. Handling of objects is delightful for them. Practical work is fascinating for them. Organisers of curriculum should not fail to suggest some practical work whenever there is a scope for it. They have rather to explore and propose opportunities of practical work. Topics which demand passive listening and abstractness are not to the taste of children, especially young ones. In their case the topics must be presented in a form which appears to arise out of their own activities. The approach should be made situational as far as possible. A topic which carries a greater scope for practical work should come in the lower classes and should gradually develop to its abstract form. It should be possible for the young child to learn every topic through practical work and real or concrete things. In case this view point is accepted, the subject matter should be arranged according to the taste of the child; further, there is absolutely no reason why a pupil of the Fourth or the Fifth class should not be allowed to make use of the foot-rule and the compass to draw figures. These figures may not necessarily conform to specified geometrical shapes. Similarly there is no reason why the children of the second class should not learn to add 3 kgms. 40 gms., and 4 kgms. 12 gms. before they learn how to multiply 18 by 19 or even to add 340 and 412.

3. THE CRITERION OF DIFFICULTY

What is either too difficult or too easy for a class should not be included in the curriculum for that class. The mental level and power of comprehension of the pupils should be taken into account while arranging the subject-matter. The organisation should suit the mental capacity and development of the pupil.

It is, of course, not easy to determine the intensity of difficulty from the pupil's point of view. He must, by all means, be offered the facility of learning from 'simple to complex.'

It is a very difficult task to decide what is really difficult and what is really easy for the child. What is simple for the mathematician may not necessarily be simple for the child. Similarly, what is difficult to a mathematician may not be so difficult for the child. Organisers will have to watch constantly the difficulties of children so that some genuine observations may be available at the time of readjustment of subject-matter.

One more theory is considered to be helpful in this regard. It is known as Culture-Epoch-Theory. The order in which the human race developed different topics of mathematics can be the best order to follow. The topics which were discovered or developed the earliest of all, must have been the easiest of all. The topics were discovered and developed in the order of their difficulty. The present day research in the subject is obviously the most difficult of all. This order will offer the path of least resistance to pupils. It, however, does not imply that the order should be followed in all cases. The criterion of utility and the psychological criterion will override this theory. When other things are equal, this theory is a real source of help in deciding the order of difficulty of topics of the subject-matter.

4. VOICE OF THE TEACHER

This point has already been adequately emphasised under the caption 'Teacher's Concern." The field worker must have a sort of final say in the selection as well as arrangement of the course content. His experience is of practical nature, so it must count in this respect. It will be very sad and harmful if this opinion is set aside.

5. TOPICAL VERSUS SPIRAL ARRANGEMENT

The criterion of difficulty emphasises that topics should be arranged in order of their difficulty. But it is no good arrangement to take the topic as a unit. Topical arrangement is based on the unity of topic. It implies complete exhaustion of a topic at the same time or at a stretch. But it is highly unnatural and impracticable to take a topic, such as time and work; in a particular class and exhaust all its possible portions and problems, simple as well as complex in the same class. Every topic has easy portions and also difficult portions. When a topic is started its elementary or introductory portions are considered to be within the comprehension of the class. At that stage the difficult portions will be certainly beyond them. Therefore the topical arrangement is defective. The chief defect is that it introduces into the curriculum many irrelevant things. Certain portions of the topics are to be imposed on the child who is not yet mentally prepared. He does not find any immediate need of, or use for, these portions. These are beyond his comprehension and so he cannot substantially gain by their study. He cannot appreciate them at that stage. They are introduced to make the subject matter of the topic complete and thorough. Moreover, in the topical system, the topics which are dealt earlier receive no attention later, and so there is every possibility of their being forgotten. There is no scope for year-to-year revision of any important topic even. Complete understanding of a mathematical topic needs a varied experience. No topic is so water-tight that other topics or subjects cannot help in its better understanding. One cannot claim that one has grasped the notion of "fraction" thoroughly before learning anything about the "unitary method."

Good arrangement requires that (a) every topic should be divided into parts, (b) those parts should be graded according to difficulty and (c) each part should be introduced at a proper stage. These different portions will be allotted to different classes according to their difficulty and the maturity of minds of pupils. This approach is known as spiral arrangement. A portion is introduced when a sufficient background has already been prepared for it. The remaining more tedious portions are left to be done later. Spiral is the opposite of topical arrangement and attempts to remove all the defect of topical arrangement. Everything is done at its proper stage, and there is an opportunity for its revision also. The only care which has to be taken about it is that the portions should be neither too long nor too short. If a portion is small, the study will last for a short period and no durable impression will be formed on the learner's mind. If a portion is very long, its study will take a very long time. The students' interest will flag; the idea will not go home.

6. THE PRINCIPLE OF CYCLIC ORDER

The number system and the four fundamental operations of arithmetic, which form the basis of mathematical development, reappear frequently in more complicated forms. This repetition of fundamental ideas in building the structure of mathematics is known as the cyclic development of the subject. It is an example of natural spiral arrangement in the subject. It is the duty of the organiser to note such possibilities of cyclic treatment of subject matter and to make adequate provision for the effective arrangement of such material. Hints to stress this cyclic repetition should be given side by side with the relevant topics in the syllabus itself.

7. Incidental Versus Organized Teaching

Knowledge grows by application. One cannot be said to have learnt anything unless one is able to use it. The application of mathematics to other subjects not only imparts clarity and definitness to them but has a healthy bearing on the learning of mathematics also The study becomes pleasant. This principle is more on the side of incidental than organised teaching. The arrangement of subject matter should promote this incidental approach in teaching.

The aim is not to teach the boy mathematics, science or language it is rather to educate him so that he may understand the real nature of life. If knowledge is divided into subjects such as history language. mathematics and so on, it is only for the sake of convenience. It is hoped that by such division the impressions of the various subjects will be synthesised and will enable the learner to form a true idea of life. But the present situation gives no promise of fulfilment of this hope. There is no cohesion. A blind faith in the theory of formal discipline is partly responsible for this situation.

This formalism has to make room for incidentalism. The student may have to learn an unspecified item of any topic on an unforeseen occasion. It is better, if such incidents are anticipated and recorded in the curriculum. In the list of activities mentioned for the students a suggestion can be given that they should collect incidents and situations which might have posed a mathematical problem before them. Thus they will be able to learn it while they are solving their practical difficulties and satisfying their needs.

8. Individual Versus Class Teaching

Curriculum has so far been organised mostly for class teaching purposes. Class teaching (though a very good compromise between efficiency and economy) is not free from defects. It has been found that there are wide differences in ability of individual pupils. Students differ in tastes, aptitudes, capacities, intelligence, interests and experiences. The class teaching system neglects individual needs and interests. It subordinates the individual to the group.

Understanding individual differences, and dealing suitably with their unique characteristics have become very important. The syllabus must help the teacher to meet this demand of emphasis on the individual. One teacher for one pupil is the ideal or one syllabus for one pupil may be taken as the ultimate goal; but initial steps should be taken in this direction. It is the teacher's responsibility to give to the quicker and brighter pupil all that he can muster and allow the slower or less intelligent pupil to proceed at his own speed. The two should not be put together. The dull may reach just the minimum while the bright may reach the maximum. Each should get opportunities to use his ability to its maximum. Both will have to be led through different routes.

One syllabus for one pupil is, of course, not practicable; but the demand of individual teaching is considerably met by the arrangement of "ability grouping". Here the students are grouped in such a way that differences in the abilities within a particular group are reduced as far as possible. There are no extremes in these groups. In a group, students possess almost the same interests and intelligence. Thus the class group will be further divided into four or more homogeneous groups. This grouping will necessitate some grouping of topics in the curriculum. The curriculum will slightly vary for different 'ability groups'. The 'Dalton Plan' has got its roots in this very approach. Different techniques to make pupils of different abilities understand the same topic may also be adopted.

It would not be out of place to mention that class teaching has got its own merits, which cannot be ignored outright. It helps the students to shake off their natural shyness and nervousness. It develops in them the power to understand, criticse and explain—a power which is greatly needed in the present social and democratic set-up of the country. Everybody is conscious of his dignity and remains alert and attentive in the classroom. Nobody likes to be in the background therefore he has to think and act quickly. When the class is engaged in work, individual help can also be given. General or common mistakes can be corrected collectively; it will save time Rules and formulae can be discovered and developed in the class. The spirit of group work and healthy competition may be created. In competition, there is always a stimulus for the students. They want to surpass one another. It enables the teacher to use a variety of illustrations and devices in order to cater to different tastes and capabilities of the individual pupils. The teacher will always be present and his presence will be a source of inspiration to all alike.

This "individual mode" has its own advantages. Everybody gets a fair chance of progress. The input and thoroughness of work increases since everybody feels keenly interested in his own work. The pupil becomes self-reliant. The teacher is looked upon as a friend in need. But there are also some disadvantages. The students miss the opportunity of group or class exercise. They will be always in an unnecessary haste and may acquire the tendency for superficial or show work. They may suspend work and sit idle just for no reason. It will be exacting for the teacher to attend to each and every individual separately.

Organisation of subject matter has to satisfy both the individual and the class interests. Some differentiated assignments may be proposed for students of different levels of intelligence. etc. There must be some provision for individual and independent work for the aspiring learner. There may be some individual projects and some group projects related to the same topic.

Thus curriculum has to devise a middle route, so that it can be adopted on class as well as individual basis. It should at least meet the requirements of the 'ability grouping' system.

9. THE DALTON PLAN

As mentioned earlier this plan suggests that curriculum organisation should accommodate the criterion of individual differences. The development of the topics should be such that pupil may be able to attempt them independently with considerable chances of success. The pupil has to depend on organised private study in this case. The syllabus will have to be broken into monthly assignments. The Dalton Plan is based on these assignments. A month's task can be further divided into four weekly instalments of work. There is no time-table, and no regular classes have to be held.

Though it is not adequately practicable to reorganise the curriculum on the lines of the Dalton Plan as the present set-up does not favour it, even partial introduction of the plan will be a healthy reform. It will give the pupil something to attempt independently.

10. THE PROJECT PLAN

According to this plan, mathematics is to be taught incidentally, when the need arises in the working out of the project. The kind of project which can form one of the bases of curriculm construction is that in which the dominating purpose is to solve a problem, to unravel a mystery, to overcome some intellectual tangle or difficulty. Such is a real project which can lead to understanding of mathematical facts. From time immemorial, it has been seen that, in schools, no good work is done under compulsion. It is the inner urge that produces good work. No good teacher will fail to use such projects to arouse and maintain interest provided these have been suggested and elaborated in the curriculum and also in the text-book. "Teach when the need arises", is a good rule but its reverse, "Don't teach unless the need arises spontaneously in the course of the pupil's work" is absurd. In the case of some boys, never may need arise for anything. "In one high school (in America) where project teaching is in vogue some of the pupils have betrayed the fact that when asked to suggest projects they always vote to take some trip in order to get away from school". "I suspect that even some of the teachers are not above this dodge", Sh. N. Kuppuswami says "under the circumstances, projects may become occasions for the pupils and teachers to learn a number of, may be, useful things but not mathematics.

In mathematics, project work, cannot supplant systematic teaching. It may lead to introduction, but cannot carry to accomplishment. Practice is so necessary in mathematics, that it can be provided by no other method than continuous sustained practice.

But the organisation of the curriculum on the project plan will certainly help the students to discover and understand mathematical facts. Dependence on project work occasionally will lend variety and life to the otherwise monotonous work. So a few suitable projects, if included in the curriculum, will not be out of place.

11. From the Empirical to the Rational

The learning of a topic in mathematics begins from experience and induction, and it is accomplished gradually through deduction. This is what is meant by the process of going from the empirical to the rational. The most familiar and evident things should come in the beginning of the topics, and general ideas or principles should be rationalized later. The subject matter should be arranged on these lines, so that the teacher can follow inductive deductive methods correctly and conveniently. From clear and evident facts to principles, then to more complicated facts and thence to more general and abstract principles—this should be the order of arrangement of the detailed portion of a topic. There should be no place for definitions in the very beginning. The syllabus should clearly conform to the steps which the teacher is expected to adopt in the class-room.

12. The Principle of Correlation

A vital question, which arises in connection with the fusion of methods of treating the subject matter, is "How can the subject matter be correlated with some other branches of mathematics and with other fields of application ?" This correlation may extend in four directions : (1) Correlation of mathematics with life, (2) Coorelation of mathematics with other subjects of the school curriculum, (3) Correlation of the different branches of mathematics among themselves, (4) Correlation of different topics of the same branch.

According to Perry, "mathematical study began because it was useful, it continues because it is useful, and it is valuable to the world because of the usefulness of its results".

Correlation With Life

One of the aims of teaching, mathematics is that outgoing students should be fully prepared for life. Even after the introduction of compulsory education, a large number of the pupils do not continue their studies beyond the primary stage. They would become labourers, farmers, petty shop-keepers or the like. They would ordinarily not come across problems like 33568×4977, $56785 \div 235$ and $\frac{8}{5}+\frac{8}{7}+\frac{4}{9}$ etc. But they may be required or may have to keep accounts in a small way, compute interest on small sums of money, find out prices of more things when the price of one thing is given, and find profit and loss. The organisers of the syllabus should keep in view the preparation of this category of students.

Pupils solve problems at school so that they may be able to solve similar problems in life. For this purpose, the problems should not only be true to life, but be put to the pupils in the forms in which they actually appear in life. The syllabus should be suggestive of such activities as require pupils to go to the bazzar and find the prices of commodities. They may collect information about whole-sale prices from newspapers etc. and compare them with the local prices. Similarly, craft activities which are a very significant part of our life, can serve as a good centre of correlation. This correlation with life is an essential element of training for citizenship. This not only creates interest, but also educates the pupil by making him understand what is going on around him.

Basic Education lays stress on correlation, but it has not been able so far to clarify what the actual form of this process of correlation is. This new idea has not made much headway, simply because its conception is not adequately clear. It has been left to the teacher to work it out as he thinks proper or feasible. It is rather a function of the organisers of the syllabus to initiate the teacher into this new technique. The concept and working of this process needs elucidation from the experts who organise or arrange the subject matter.

CORRELATION WITH OTHER SUBJECTS

To the student of science and related fields, the various applications of mathematics constitute the most interesting phase of correlation. He needs the applications of geometry, algebra and trigonometry. The organisers should recommend various topics which may be useful in the proper study of other subjects. This type of correlation saves time and also enables the student to do the work more efficiently and effectively. As far as possible, connected topics of different subjects should be suggested for the same year, even for the same occasion.

Other subjects can also serve as sources for supplying exercises and problems in mathematics. The organisers should draw upon them unhesitatingly. To facilitate correlation, the organisers should mention the connected topics of other subjects along with the topics of the syllabus of mathematics.

CORRELATION OF DIFFERENT BRANCHES OF MATHEMATICS

It implies that there should be correlation of these branches among themselves. Shri N. Kuppuswami emphasises the need of this correlation, "Only the other day I found that in a class, that had learnt to find the cost of carpeting the floor, gravelling paths, and plastering the four walls of a room, the boys did not know what a rectangle was much less how to draw one. The teacher very gravely explained that they had not come to that stage in practical geometry". Similary there are many teachers who do not use equations in solving arithmetical problems. Nobody can refute the arguments that algebraic method is a far better method in solving many typical problems in arithmetic. It is contrary to the very purpose of mathematical education to ignore an easy and general method for the sake of following difficult methods applicable only to particular problems. The equation method is the most natural method of solving difficult arithmetical problems. This method saves time which can be utilised in other ways.

The compartmentalisation of the subject into different branches has outlived its utility. In schools the idea of teaching them separately has to be given up. There will be occasions when correlation between algebra-geometry, arithmetic-algebra and arithmetic-geometry will facilitate the teacher's and the pupil's work.

The task of separating the branches was very easy, but now coming back to their cohesion needs a thorough planning and sustained action. The Organiser of the syllabus can help a good deal in this respect. The topics which are inter linked, should be grouped. They should come together. Their common factors should be given a vivid treatment in the syllabus. Whenever one formula or idea is taken up in one branch, it should be possible to tackie it in other branches also. Even addition, subtraction and multiplication problems of algebra can come simultaneously with allied arithmetical problems.

CORRELATION OF TOPICS IN THE SAME BRANCH

There must be a logical sequence or continuity in the different topics of the same branch. Each topic in whatever branch it may be, should show its bearing on its preceding topics and succeeding topics. The different units should be so correlated as to bring out clearly the object of teaching the whole subject.

It has been emphasised in a preceding paragraph that the branches should be merged into one. It should not be taken to mean that the present stand contradicts the previous one. Branches or no branches, inter-topic correlation is a "must". Even if all branches are merged to form a new single branch the topics in it will have to be arranged in a sequence.

The organisation of topics in a branch should be perfectly logical so that every topic grows from previous topic and leads to the next topic in the particular case.

PROBLEMS

A problem is an exercise and something novel. Sometimes it is compared with a project. Mathematics differs from other school subjects, as it provides an opportunity of solving problems which evoke thinking. Training children to solve problems is meant for training them to meet and surmount difficulties, and for enabling them to solve problems offered by life itself. But the proposed approach to the nature and scope of problems has been misdirected to some extent. The problems so far framed and proposed have served no better purpose than making trapped-in-the-school boys solve them.

The problem is the next step after the syllabus, in the educational process in mathematics. Suitability of the problem is very important. Generally the text-book writers have followed the traditional line in this respect. Some problems have become a "must" in every text-book, and those are very important also for examination purposes. Since the text book writers have failed to deliver the goods, therefore the organiser has to show them the way. The syllabus should contain model problems on its topics and text-book must be written accordingly. The text-book writers should be obliged to come out of the old rut. Every problem should be a novelty and should pose a new challenge to the thinking capacity of the pupil.

While judging the quality of a problem one should be very clear about the following points :—

1. No problem should be made to look real when it is in fact incredible and unrealistic.

2. The difficulty of a problem or its capacity to challenge intellect is not a sufficient reason why everybody should solve it. The problem should have something in it which may necessitate its solution from the child's point of view.

3. Similarly it is not enough that a problem gives the mind a chance to reason. The problem should be worth reasoning about. The reasoning involved in it should be genuine.

4. That a problem gives an opportunity to apply a mathematical principle or formula, is not in itself a good and adequate reason for its inclusion in the subject-matter. The school problems should be of the same kind, and should be put under the same circumstances, as those offered by life itself.

Here is a survey of the different types of problems available in mathematics.

1. PUZZLE PROBLEMS

Puzzle problems are meant to bewilder and perplex the student. These are mostly solved as a pastime and recreation. Their solution demands patience, but their intricacy helps in maintaining the curiosity of the student. But puzzle problems have been excluded from the current subject matter. Their inclusion in small numbers will however be a healthy change. The boy who has done his day's work may, as a reward, have his recreation in the form of solving such problems. The problems in "Lilavati" were framed, we are told, to serve as a pastime or recreation for the widowed daughter of the author. These problems are useful in another way also as they offer an opportunity of drill in the mechanical operations of mathematics. Some young students will certainly show more interest in this type of problems than in any other type. Their interest and skill in mathematics will have to be developed through puzzle problems.

A few examples of the puzzle problems are :—

(i) Magic square : —Complete a magic square of nine places, with number 1—9, so that the sum of the numbers in each row is 15.

(ii) An example from Lilavati :—"The square root of half the number of bees in a swarm has flown out on a jessamine-bush, of the whole swarm has remained behind; one female bee flies about a male that is buzzing within a lotus-flower into which he was allured in the night by its sweet odour, but is now imprisoned in it. Tell me the number of bees ?" Answer 72.

(iii) A snail climbs up at the rate of 2 feet per hour throughout the day and falls down half the distance in the night. How much time will it take to reach the top of a pillar 60 ft. high? (Five days).

2. CATCH PROBLEMS

These are remarkable for the jugglery of words. In a puzzle problem the pupil knows its nature. He understands that his power of thinking is being tested through the intricate problem. But in a catch problem, he is not sure about its exact nature. Here the purpose is to test the mental alertness of the pupil. The pupil will attempt it correctly if he can catch the meanings of the particular words involved. A puzzle problem is generally long and difficult, whereas a catch problem is generally short and not so difficult. Catch problems are also good for mental gymnastics and amusement. There is no reason why the pupils should not have some practice in such problems at school. When used occasionally, they are likely to serve a useful purpose.

A few examples of the catch problems are :—

(i) You have to give a pillar after every three feet, how many pillars will have to be erected to support a verandah 30 feet long. (Answer 11)

(ii) A dying man wills that his horses be divided among his three sons that the eldets gets half of them, the younger gets one third of them and the youngest one ninth of them. He leaves behind 17 horses. How should the horses be divided to satisfy the three sons? [Add one horse to 17 to make 18 and perform division. They get 9, 6 and 2 respectively.]

(iii) Sohan has a double number of balls than Mohan. If Sohan has three balls more than Mohan, what is the total number of balls with them? (Nine).

3. UNREAL PROBLEMS

These are problems which are not true to the real situations of life. They create wrong notions in the minds of the learners and are harmful to them. Their existence in the subject matter is without any sound basis. The information contained in them is false and misleading. It is difficult for the young child to understand and appreciate them.

A few examples of the unreal problems are :

(i) Three pipes can respectively fill a tank in 8, 10 and 12 hours, and a fourth pipe can empty it in 6 hours. All the four pipes are opened. How much time will it take the tank to fill ?

(ii) Find out the least number which when divided by 48, 64, 72, 80, 120 and 140, leaves respectively the remainders 38, 54, 62, 70, 110 and 130 ?

(iii) The price of three tables is the same as that of two chairs and that of five chairs is the same as that of three trunks. If eight trunks cost Rs. 1240.00, find the price of one table ?

4. REAL PROBLEMS

These are problems which are true to life and are directly related to the actual life situations. These provide a real training for life and prepare the students to face various new and even unexpected situations in different walks of life. It is urgent that such problems be allotted a major share in the subject matter. The students will develop a natural affinity for real problems. They are very useful for the purposes of correlation, activity, project work and understanding. After adequate practice in them, the students will enter life with confidence.

A few examples of the real problems are :—

(i) You have to purchase a table cloth for your table 3 ft. × $2\frac{1}{2}$ ft., and you want that it should fall by six inches on all sides. What should be the length and breadth of your table cloth ?

(ii) The rate of income tax is 2%. After paying his income tax, a person is left with Rs. 4700%. If the rate of tax increases by 1%. what will be his real income ?

Similarly there can be many more examples to illustrate this type of problems.

ORAL AND WRITTEN FORMS

All the four types of problems can be set both in oral and written forms. The subject matter should contain both these forms in proper proportions. To enable the teacher to proceed from simple to complex in the matter of problems, there must be some provision of simple or oral problems. In a way, these are essential at the introductory stage of each and every topic. The progress should gradually be made from oral problems to written problems. The difficulty of the problems should increase by degrees.

A few general suggestions about selection and formulation of problems :

Unreal and undesirable problems at present occupy a dominating position in the subject matter. The theory of formal discipline and transfer of training has been stressed too far to justify the existence of this rejected stuff. This cannot be allowed to continue. The undue emphasis has to be righted. Following points may be borne in mind while determining the worth of a problem :—

1. It should be real.
2. It should have social value.
3. It should have environmental value.
4. It should occur in everyday activities of the pupils.
5. It should be helpful in other subjects or it may have been borrowed from another subject.
6. It should promote the achievement of general aims of education.
7. It should have been borrowed from business, industry or some other life situation.
8. Its intellectual discipline should be transferable.
9. It should form an indispensable basis for higher studies.
10. It should challenge the intellect and be otherwise rational.
11. It should be interesting and enjoyable.

12. It may be put in the form of a story associated with a child.
13. It should have some historical background or significance.

EXERCISES

1. What is the importance of the knowledge of organisation of the subject-matter in mathematics to an ordinary teacher of the subject ?
2. Enlist and elaborate the important principles of the organisation of the syllabus of mathematics ?
3. Who will organise the syllabus of mathematics and how ?
4. What type of problems do you propose to be included in the subject-matter of mathematics ?

S.C.E.R.T., West Bengal
Library
Calcutta
25/3, B. T. Road

CHAPTER SEVEN

METHODS OF TEACHING

'HOW TO TEACH' is a really difficult problem for the teacher. Teaching, as it is generally said, is an art. Methods are the ways to understand and practise the art. 'Why' and 'what' of mathematics have so far been discussed, and this chapter deals with the 'how' of mathematics. "How to impart its knowledge ? How to enable the child to learn, it" are the questions to be answered in this discussion. It is the final step of the execution of what we plan to teach in mathematics.

Different methods of teaching have been proposed or propounded by different educational thinkers or schools of thought in education. It is but desirable for the student to know about all of them, so that he can make a rational choice for himself. The knowledge of procedures, merits and demerits of all the methods will broaden the outlook of a would-be teacher. The choice for him is not to be made narrow. It should be then left for him to decide form his wide information, which of the methods to use and when.

The following methods have been discussed in detail :

1. Lecture Method
2. Dogmatic Method
3. Inductive-Deductive Method
4. Heuristic Method
5. Analytic-Synthetic Method
6. Laboratory Method
7. Project Method
8. Topical Method
9. Concentric Method

LECTURE METHOD

DEFINITION

It is the method of presenting the word picture of an idea; or the method of imparting information through a speech. Lecture is another name for a speech, and when you are speaking continuously to a class or an audience, you are considered to be lecturing. It is the method of depicting everything in words. The speaker speaks and the listeners listen. It is a one way traffic of flow of ideas ; the speaker gives ideas and the listener takes them.

PROCEDURE

How to apply this method to the teaching of mathematics ? What is the shape or form of this method ? The teacher prepares his talk at home and pours it out in the class. The students sit silently, listen attentively and try to catch the point. Supposing "Profit and Loss" is the topic in hand. The teacher goes on telling and explaining "Well boys, profit and loss is always to be calculated on the cost price, because the cost price is our investment in the bargain if you invest less and earn more you gain; therefore gain is to be calculated by subtracting cost price from the selling price. When you invest more and earn less, you lose; therefore loss is to be calculated by subtracting selling price from the cost price, so on and so forth." If the teacher is extremist in lecturing he may not even write anything on the black board simultaneously or he may not even argue a point with the listeners by cross questioning. The method takes the form of a "One man's show", where the listeners remain passive.

ITS ADVANTAGES

1. When the number of students in class is very large, this method is the only way out. The teacher's voice is heard clearly even in the farthest corner of the class-room. All the students are provided with an equal opportunity to listen and learn.
2 When a heavy syllabus is to be covered in a short time, lecture method is suitable. The teacher can teach at his own speed and thus finish a topic in one or two days as he likes. He need not adjust his teaching to the learning speed of the students.
3. The situation becomes impressive. When the teacher is delivering his lecture frequently, the students are listening attentively and there is pin drop silence among them. An outsider forms a good impression of the discipline in the school.
4. The method is convenient for the teacher, because he does not have to give individual help.
5. There is always a natural sequence between the ideas of a topic and this sequence or interconnection can be best maintained in a lecture. Sequence of ideas in a carefully planned lecture makes the topic understandable. The present idea follows from the previous idea and leads to the next idea. This interrelationship of ideas facilitates learning.

6. The teacher and the taught feel satisfied at their respective places. After delivering a lecture the teacher feels satisfied that he has finished a part of the syllabus. After listening to a lecture the student feels satisfied that something new has been taught to him. Such satisfaction may be denounced, as false but it is there.
7. In certain topics like insurance, budget, interest and income tax, an introductory talk may be found most impressive and useful.

ITS DISADVANTAGES

These can as well be interpreted as demerits of the method.

1. The method gives a false sense of satisfaction, which is dangerous and harmful.
2. Receiving and memorising bits of information is not so important a purpose of the study of mathematics as it is thought to be. Rather, its study is more concerned with the thinking and reasoning power of the learner.
3. The students may remain inattentive during the lecture.
4. They remain passive.
5. Experimentation is totally neglected. There is no opportunity for the students to discover and find out facts for themselves.
6. There is no scope for the development of close contacts between the teacher and the taught. The teacher can't give individual guidance because he can't properly judge the difficulties and potentialities of an individual.
7. There is a rapid and hurried flow of ideas in a lecture. Everything may not be comprehended by young learners.
8. If somehow or other, a student fails to catch any essential point in a lecture, he won't be able to understand the subsequent ideas, because the ideas follow logically.
9. It is a sort of putting everything from above, where different abilities and capacities of the child are not made use of.
10. Homework is likely to be very heavy.

CONCLUSION

The reader must have noted that the method suits neither the subject nor the learner. It is an informational method which goes against, independent and original thinking of the learner. There is no student participation in the learning process. Very few mathematical topic lend themselves to effective treatment by this method. The teacher talks continuously. Most of the time his face is towards the class, and the back is towards the blackbord. This is defective. He should mostly face the blackboard. It must, however be conceded that, on topics like income tax, and insurance, lectures by experts from outside may be arranged and would be found useful. But lecture method is highly unsuitable as a regular method in classwork.

DOGMATIC METHOD

In this method of teaching mathematics rigour is extremely emphasised 'rigour' means 'the strict enforcement or observance of rules. The dogmatists say that the foremost educational value of mathematics is the training in exactness which it amply provides. Mathematical knowledge observes a high standard of exactness. Any deviation or departure from this standard of exactness will defeat the very purpose of teaching mathematics, and will consequently lead to inexact, aimless and slipshod thinking. The advocates of this method say that the inefficiency of mathematical teaching is mainly due to lack of rigour.

PROCEDURE

The rules and formulae are given to the class to cram. The teacher tells the pupils what to do, what to observe, how to attempt and how to conclude. He works out the model sums on the blackboard and the pupils have merely to follow the patterns. The steps of the solution of a problem are brought home to the students who then follow them in their minutest details. The model or pattern as presented and advocated by the teacher or the book is to be strictly adopted and imitated by the learner.

MERITS OF DOGMATIC METHOD

1. The method saves time, energy and a good deal of loose or useless thinking.
2. It can be adopted with advantage at a stage when pupils are adequately advanced in mental development.
3. It promotes skill, efficiency and speed in the solution of problems.
4. It glorifies memory.
5, At the revision or application stage emphasis on rigour is most desirable and appropriate.

ITS DRAWBACKS

1. In this method rules and formulae form the beginning of knowledge, whereas these are truly the ends of knowledge.

2. The stress is laid on rigour, and no attention is paid to the specific abilities of students. Mechanical cramming is enforced without any regard for the individual's capacity.
3. There is no emphasis on development of power to acquire and apply knowledge, independent thinking and originality; so the real aim of teaching Mathematics is ignored.
4. The subject becomes dull and uninteresting for the students. Mechanical methods cannot promote interest.
5. Most of the mechanically memorised information is likely to be forgotten soon.
6. It is the method of imparting information, which is a wrong approach to the teaching of the subject.
7. If mechanically memorised facts just slip from the memory at the time of need, the student fails to recall or reconstruct them.
8. The student repeats and adopts others ideas. He becomes a slave of the ideas of others.
9. Students without real mathematical ability, but with good cramming capacity, begin to consider themselves successful students of the subject, which is a dangerous notion.

CONCLUSION

This method suits neither the child nor the subject. The mind of the student is stuffed with information, and the understanding of the subject finds no place there. Therefore, if popularised, this method will cause stagnation in teaching. Still it has one commendable aspect, that is, its spirit should influence the student when he is applying or revising knowledge and when he is quite mature and advanced in the study of this subject. Errors, incompleteness, inexactness and vagueness should not be allowed and overlooked in these cases. If the student commits these mistakes even at the revision stage, or when he is expected to be adequately mature, when shall he be able to do things correctly ?

INDUCTIVE-DEDUCTIVE METHODS

It is a combination of two methods. To be able to understand this combination, one shall have to understand them separately.

INDUCTIVE METHOD

It leads from concrete to abstract, particular to general and from examples to general rule. It is the method of constructing a formula with the help of a sufficient number of concrete examples. It is based on induction which means proving a universal truth by showing that if it is true for a particular case and is further true for a reasonably adequate number of cases, it is true for all such cases. A formula or generalisation is thus arrived at through a convincing process of reasoning and solving of problems. After a number of concrete cases have been understood, the student successfuly attempts the generalisation.

PROCEDURE

Ex. 1. Ask students to draw a few sets of parallel lines with two lines in each set. Let them construct and measure the alternate and corresponding angles in each case. They will find them equal in all the cases. This conclusion in a good number of cases, will enable them to formulate the relevant generalisation.

Ex. 2. Ask them to construct a few triangles. Let them measure and sum up the angles in each case. The sum will be the same in all the cases. Thus they can safely conclude that the sum of angles of a triangle is equal to two right angles.

Ex. 3. Give them a number of cases e.g., $a+b$, $x+y$, $l+m$ and $p+q$, and ask them to find the squares in each case, by the method of multiplication. They can further be helped to generalise on the basis of these conclusions that

$$(\text{1st term}+\text{2nd term})^2=(\text{1st term})^2+(\text{2nd term})^2 +2(\text{1st term})\times(\text{2nd term}).$$

Ex. 4. Give 3, 5 and 7 things to three students individually, and then ask them to divide the things equally among themselves. For this purpose they will first calculate the total number of things and then divide the total number by the number of students. Again give 3, 6, 7 and 8 things to four students individually. To divide the things equally among themselves, they will first calculate the total number of things and then divide this total by the number of students. Another similar concrete case may be taken. The generalisation can be made on these cases. It may be introduced to them that this equal quality is known as the average. To calculate this average they have to first calcualate the sum of the given quantities, and then to divide this sum by the number of qualities. Thus the following formula or generalization is arrived at :

$$\text{Average}=\frac{\text{Sum of the terms}}{\text{No. of terms}}$$

There can be many more examples to illustrate the procedure to be followed in this method.

MERITS OF THE INDUCTIVE METHOD

1. It helps understanding. It is easy to understand a mathematical

principle established through a number of simple examples. Any doubts about the "how and why" of a formula are clarified in the very beginning.

2. It is logical method. So it suits mathematics.
3. It gives the opportunity of active participation to students in the discovery of formula.
4. It is based on actual observation, thinking and experimentation.
5. It curbs the tendency to learn things by rote, and also reduces homework.
6. As it gives freedom from doubts and helps in understanding, it suits the child.

DRAWBACKS OF THE INDUCTIVE METHOD

1. It is limited in range. It contains the process of discovering the formula with the help of a sufficient number of cases, but "what next?", is not provided in it. The discovery of a formula does not complete the study of the topic. A lot of supplementary work and practice is needed to fix the topic in the mind of the learner.
2. Inductive reasoning is not absolutely conclusive. Three or four cases are picked up to generalise an observation. Therefore the process establishes a certain degree of probability which can, of course, be increased and made more valid by increasing the number of cases.
3. It is likely to be more laborious and time consuming.
4. At the advanced stage, it is not so useful as some of the unnecessary details and explanation may make teaching dull and boring.
5. Its application has to be restricted and confined to understanding of rules in the early stage. Once a formula has been established, time should not be wasted in rediscovering it for every subsequent problem.

DEDUCTIVE METHOD

It is the opposite of Inductive Method. Here the learner proceeds from general to particular, abstract to concrete, and formula to examples. A pre-construed formula is told to the student and they are asked to solve the relevant problems with the help of that formula. The formula is accepted by the learners as a pre-established and well-established truth.

PROCEDURE

Immediately after announcing the topic for the day, the teacher gives the relevant formula. To explain further the application of the formula to problems, he solves a number of problems on the blackboard. The students come to understand how the formula can be used or applied. Then a few problems are given to the students. They solve them on the same lines as have been explained by the teacher.

Area of a rectangle$=$Length$\times$Breadth.

$$\text{Simple interest}=\frac{\text{Principal}\times\text{Rate}\times\text{Time}}{100};$$

and profit or loss is always calculated on the cost price. The students apply these formulae to solve the problems, and then memorise them for future use.

MERITS OF THE DEDUCTIVE METHOD

1. It is short and time-saving. The solving of problems by predetermined formulae takes little time. Authors and teachers, therefore like to adopt it (give it preference over others).
2. It glorifies memory, as students have to memorise a considerable number of formulae.
3. At the "practice and revision" stage, this method is adequate and advantageous.
4. It combines, with the inductive method to remove the incompleteness and inadequacy of the latter.
5. It enhances speed and efficiency in solving problems.

Its Drawbacks :

1. It is very difficult for a beginner to understand an abstract formula if it is not preceded by a number of concrete instances.
2. Pure deductive work requires a formula for every type of problems and an extensive use of this method will demand blind memorisation of a large number of formulae.
3. It will thus cause an unnecessary and heavy burden on the brain. It may even result in brain fag.
4. Memory becomes more important than understanding and intelligence, and that is educationally unsound.
5. If the pupil forgets the memorised formula, which is very likely to happen in case of blind cramming, he is at a loss and cannot recollect and reconstruct the formula easily.
6. The students cannot become active learners.

7. It is not suitable for the development of thinking, reasoning and discovery.

Conclusion :—The discussion of the scope, merits and drawbacks of these two methods leads us to conclude that Inductive Method is the forerunner or predecessor of Deductive Method. The Deductive Method will give a good follow-up, if it is preceded by understanding through induction. Any loss of time due to the slow speed of induction can be made up through the quick and time saving process of deduction. There may be a number of arguments against deduction but it cannot be driven out of the field. It is to serve as the complement of induction. Induction leaves the learner at a point where he cannot stop; the after-work has to be completed by deduction. Deduction is a handmaid of induction; and what is left undone in induction, is fulfilled by deduction. The two methods are such good partners that the shortcomings of the one are offset by the other. Deduction is a process peculiarly suitable for a final statement, and induction is most suitable for the exploration of new fields. Deduction is the chief glory of mathematicians, but they take their facts from induction. Probability in induction is raised to certainty in deduction. The happy combination of the two is most appropriate and desirable. There are two clear-cut major parts of the process of learning of a topic; viz, establishment of formula, and application of that formula. The former is the work of induction, and the latter the work of deduction. Mathematics in the making is inductive, and its finished form is deductive. Understand it inductively and apply deductively. Application and practice should be preceded by understanding. Blind practice leads one nowhere. Thus the teaching should begin with induction and end in deduction. Induction gives the lead, deduction follows.

HEURISTIC METHOD

The term 'Heuristic' is derived from a Greek word, which means 'I find'. Here the child is put in the place of a discoverer. The method involves finding out by the student, instead of merely telling of everything by the teacher. It aims at removing the shortcomings attributed to lecture method. Contrary to lecture method, it demands complete self-activity or self-education on the part of the learner. It is a method by which pupils learn to reason for themselves. Staunch supporters of this method are of the opinion that every child should be made a discoverer and inventor. When this method is applied in its extreme form, the teacher stands aside as an onlooker and the child selects his own path and proceeds according to his own lights. The teacher is not required to encourage, help or guide him. There is no need of the teacher's approval or disapproval of his work. If he wants to accomplish something, let him do it in his own way. The teacher should not impose or thrust his reasoning, argument and pro-

product of thinking on him. Let the child help himself with reasoning and argument. The bookish, readymade and teacher-made arguments should not be forced on him. This method makes the student self-reliant and independent. It imparts a sound training in the education of self. The teacher's job is not to slove problems for the pupil, but to enable the pupil to solve problems for himself.

Prof. Armstrong was the originator of this method. He devised it for the teaching of science. He was of the opinion that the school should develop thinking power of student, and for this the latter must be given an opportunity to think. This method has been found useful in the teaching of mathematics also. It is a sort of attempt to develop in the learner a particular attitude, now popularly known as the scientific and heuristic attitude. It aims at training a learner in the method of learning. Knowledge is not the primary consideration here. Self confidence originality, independence of judgement and thinking power are to be developed in the individual to make him an ever successful student.

Practically speaking, it may be well-nigh impossible to adopt the extreme form of this method—the form where the teacher is simply a passive observer and the child has to learn in his own manner. The is child after all a child, he is ignorant, most of the things are new to him. He needs guidance, and in certain cases he may need the teacher's help at every step. He needs time and help to develop proper acquaintance with the new knowledge. Some hints or instructions have to be given when he does not show continuous and steady progress. He should overcome his difficulties by his own efforts as far as possible, but the teacher's help has not to be denied or withheld when he needs it badly. Teaching has to become a process of the maximum possible participation of the child.

PROCEDURE

Ex. 1. Take the problem of the discovery of the characteristics of a parallelogram. Give students many parallelograms already drawn on sheets of paper and ask them to find out the qualities of the different elements of a ∥ gm. Naturally the students will start judging and measuring the elements of different ∥ gms. their findings about the equality of its opposite sides and the equality of the sum of its adjacent angles, will perfectly agree. Thus their respective observations will enable them to generalise about some of the characteristics of a ∥ gm. They can further be encouraged to draw their diagonals and find out after actual measurements that these bisect each other in every parallelogram.

Ex. 2. Take the problem of the area of the four walls of their classroom. The teacher should remember that he has to educe the process from the learners. By carefully framed and graded questions he has to lead the students to discovery—e.g. How is the area of all the four walls found ?

(First shall be found the area of each separate wall, excluding the area of windows, ventilators or doors in it; and then the sum of their areas will be easy to calculate.)

What is the form of each wall ? (It is rectangular.)

[By beginning from one side.] How to find the area of this wall ? (By multiplying its length and height.)

Its length = 25 ft. and height = 12 ft. therefore its area = 25 × 12 square feet or 300 sq. ft.

Similarly questions regarding each separate wall have to be put. By appropriate questions the students would be made to observe that the opposite walls have the same areas, and so the double of the area of one gives the area of two opposite walls. By a few more similar problems the learners would be made to realise that there can also be a formula which will be applicable to the areas of four walls of any room. Questioning has to replace telling in the classroom.

MERITS OF THE HEURISTIC METHOD

1. The student becomes an active participant in the learning process.
2. The student thinks for himself and does not merely listen for information.
3. Home study and memorisation work become light.
4. It is certainly a psychologically sound method, as it aims at utilising the active, original, creative and constructive tendencies of the learner.
5. After discovering something by his own efforts, the student starts taking pride in his achievement. It gives him happiness and mental satisfaction, and encourages him towards further achievement.
6. The student acquires a real understanding and clear notion of the subject. It gives him a complete mastery of what he has learnt.
7. The teacher remains in constant touch with his students.
8. It develops in them the heuristic attitude or scientific attitude.
9. It creates in them a spirit of enquiry.
10. The student becomes self-reliant.

Its Drawbacks :

1. It demands extraordinary labour and special preparation

from the teacher, who is already over-burdened.

2. Every teacher may not be able to use it successfully. The teacher must be gifted with the heuristic spirit.
3. It is a slow method. Too much time is taken up by investigation.
4. There is very little certainty that, the child left to himself will make steady and sufficient progress.
5. Every child cannot be expected to be a gifted discoverer. The immature child has his limitations and difficulties.
6. Specially, in early stages, the child needs guidance and hints. If the teacher does not give him proper guidance, he may get discouraged and disgusted.
7. The teacher has to give well-measured guidance, neither more nor less. Over-guidance may also harm the initiative of the learner and make him dependent.
8. If the students are tempted to consult books and copy from them for the sake of discovery, the method fails in its purpose. Imagine the following situation. The teacher announces, 'Today we shall discover such and such formula', and a student gets up in his seat to relate the formula word by word.
9. It may not be possible to teach all topics by this method. In some cases, discovery may not be possible at all.
10. The method presumes small classes, because it demands individual attention on the part of the teacher.
11. Sometimes the teacher cannot frame good questions. Bad questions leave little to the imagination of the child, and do not provoke real thinking.

12. Sometimes the teacher fails to distinguish between *false heuristic and true heuristic.* Then the method degenerates into worthless questioning. The difference is illustrated below :

False heuristic	*True heuristic*
1. Is it true that a square has all sides equal ?	1. What do you know about the sides of a square ?
2. Do you remember that area of a rect. = length × breadth ?	2. How do we calculate the area of a rectangle ?
3. Tell me whether the profit or loss % is calculated on the cost or selling price ?	3. On which price do we calculate profit or loss % ?

CONCLUSION

At school, use of the extreme form of this method is out of the question. The teacher's presence in the classroom should mean something. He is not to behave as an indifferent onlooker, but his presence is to inspire and stimulate the learners. In practice, the success of this method depends largely on good questioning. It is the spirit behind this method, which matters most. Nothing has to be told, so long as the child is progressing by himself. His limitations and difficulties, of course, cannot be allowed to baffle him. The teacher no longer teaches; he guides. The learner no longer listens; he finds. It is, in reality, a scientific and psychological method of learning. The teacher should frame the hints and instructions very carefully to avoid over-feeding and under-feeding. He should let the child be his own teacher, and also see that his difficulties are removed in time. Whatever be his method of teaching, the guiding principle should be the adoption of the heuristic approach.

ANALYTIC AND SYNTHETIC METHODS

These two methods are applicable in combination. To conclude anything about them, one has to understand them separately first.

ANALYTIC METHOD

It proceeds from unknown to known. 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. It is the process of unfolding of the problem or of conducting its operation to know its hidden aspects. Start with what is to be found out. Then think of further steps and possibilities which may connect the unknown with the known and find out the desired result. In its original sense the verb "to analyse" means to loosen or separate things that are together. About analysis, Thorndike says that all the highest intellectual performance of the mind is analysis.

PROCEDURE

Ex. 1. Here is a proposition : "If from the centre of a circle, a a st. line be drawn to bisect a chord, it will be perpendicular to the chord'. You shall have first to perform the construction as provided in the statement. Then the analysis of the statement will be performed. What is given ? (The st. line OC from the centre bisects AB, such that $AC=BC$).
What is unknown ? (OC is $\perp$ AB or $\angle OCA = 90^0$).

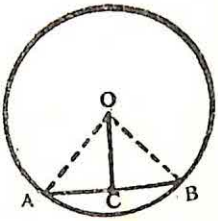


Fig. 1

Now is the stage to make a beginning with the analytical proof. How can you show $\angle OCA = 90°$? (We shall have to prove it equal to its adjacent angle. So prove $\angle OCA = \angle OCB$.)

How can you prove the equality of two angles ? (By proving the congruency of two triangles which contain those angles.)

Where are the triangles ? (Not available in the diagram, therefore we have to construct them).

Which is the possible construction ? (Const. : Join OA and OB)

Now, which are the two triangles ?

($\triangle$s AOC and BOC).

How to prove their congruency ? or Do you find some equalities in the elements of these triangles to enable you to prove their congruency ? [Yes, $AC = BC$ (given) $OC = OC$ (being common) and $OA = OB$ (radii of the same circle)]

Hence the congruency of the two $\triangle$s is proved.

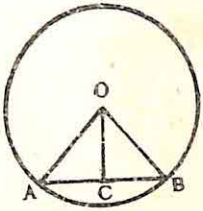


Fig. 2

What is the conclusion ? ($\angle OCA = \angle OCB = 90^\circ$).

$\therefore$ OC is $\perp$ AB.

Ex 2. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{ac-2b^2}{b} = \frac{c^2-2bd}{d}$.

The analysis will start from the unknown part of the statement.

$\frac{ac-2b^2}{b} = \frac{c^2-2bd}{d}$ is to be proved true.

What should be the first step towards the simplification of these two sides of the equation ? (Cross-multiplication).

$\therefore \quad \frac{ac-2b^2}{q} = \frac{c^3-2bd}{d}$ will be true

if $acd-2b^2d = bc^2-2b^2d$

What is the next possibility of further simplification ?

(Cancelling $-2b^2d$, which exists on both sides of the equation).

$\therefore \quad acd-2b^2d = bc^2-2b^2d$ will be true

if $acd = bc^2$.

What next ? ('c' can further be cancelled as common on both sides.)

$\therefore \quad acd = bc^2$ will be true if $ad = bc$.

Arrange it now in a more systematic form.

$ad = bc$ will be true if $\frac{a}{b} = \frac{c}{d}$, which is known and true.

Therefore by going back through the chain of argument, you can say that $\frac{ac-2b^2}{b} = \frac{c^2-2bd}{d}$ is also true.

Ex. 3. In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing the acute angle minus twice the rectangle contained by one of these sides and the projection upon it of the other.

What is given here ?

Given : An actue $\angle$d $\triangle ABC$, acute $\angle$d at B.

What has to be proved ? or What is unknown ?

(Note :—The second part of the statement may be read more than once.)

Which is the side opposite to the acute $\angle B$? (It is AC.)

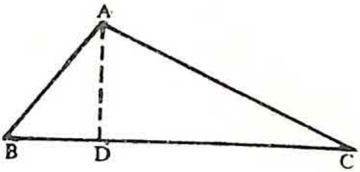


Fig. 3

$\therefore$ Square on this opposite side is AC^2.

Which are the sides containing this acute $\angle B$? (AB, BC)

Sum of the squares on these sides is $AB^2 \times BC^2$.

What more is needed ? (Twice the rect. contained by one of these sides and the projection upon it of the other.)

But, where is the projection ?

Not available in the diagram, therefore it has to be constructed.

Construction :—Draw AD $\perp$ BC, so that BD is the projection of AB on BC.

Now which are the two sides of the rectangle ?

(BC and BD).

What is the twice of this rectangle ? ($2BC \times BD$).

$\therefore$ To prove : $AC^2 = AB^2 + BC^2 - 2BC\ BD$.

Proof : It naturally has to start from the unknown.

$AC^2 = AB^2 + BC^2 - 2BC.BD$. will be true

(Try to convert the L.H.S. and R.H.S. of the equation into some known elements.)

If $AC^2 = AD^2 + BD^2 + (BD+CD)^2 - 2BC.\ BD$.

($\because AB^2 = AD^2 + BD^2$ and $BC = BD + CD$)

(Simplify the R.H.S. further.)

or if $AC^2 = AD^2 + BD^2 + BD^2 + CD^2 + 2BD.\ CD - 2BC.\ BD$.

Write down the value of AC^2 from the triangle.

($AC^2 = AD^2 + CD^2$).

$\therefore$ The statement will be true

$AD^2+CD^2=AD^2+CD^2+2BD^2+2BD.\ CD-2BC.\ BD$

(What do you notice about further possible simplification ?)

(If the last three terms on the r.h.s. cancel out, the problem is solved.)

If $AD^2+CD^2=AD^2+CD^2+2BD\ (BD+CD)-2BC.BD$ is true
or if $AD^2+CD^2=AD^2+CD^2+2BD.BC-2BC.BD$ is true.

What can you cancel now ? $(2BD.BC-2BC.BD)$
that is, the original statement will be true
if $AD^2+CD^2=AD^2+CD^2$ which is obviously true

∴ Reversing the argument,
$AC^2=AB^2+BC^2-2BC.BD.$ Q.E.D.

MERITS OF THE ANALYTIC METHOD

1. It is a logical method. It leaves no doubts and convinces the learner.
2. It facilitates understanding. It also strengthens the urge to discover facts.
3. The steps in its procedure are developed in a general manner. No cramming of a fixed step and a set pattern is necessitated. Each step has its reason and justification.
4. The student is throughout faced with such questions as "How to prove the equality of two angles?" "How to simplify the two sides of an equation ?" "What are the possible ways of resolving a statement into simpler elements etc. ?" Thus the student grapples with the problem confidently and intelligently. He gains in comprehension and skill.

Drawbacks

1. It is a lengthy method.
2. With this method, it is difficult to acquire efficiency and speed.
3. It may not be applicable to all topics equally well.

SYNTHETIC METHOD

It is the opposite of the analytic method. Here one proceeds from known to unknown. In practice, synthesis is the complement of analysis. To synthesise is to place together things that are apart. It starts with something already known and connects that with the unknown part of the statement. It starts with the data available or known and connects the same with the conclusion. It is the process of putting together known bits of information to reach the point where unknown information becomes obvious and true.

PROCEDURE

As the comparative study of these two methods is very significant, it will be more appropriate to illustrate synthesis with the help of the same problems as have been used in the case of analysis :

Ex. I. If from the centre of a circle, a st. line be drawn to bisect a chord, it will be perpendicular to the chord.

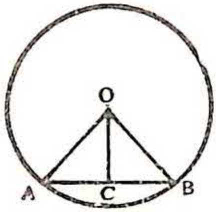


Fig. 4

Synthetic proof :—

Given : In the circle, OC bisects AB, such that AC=BC.

To prove : $<OCA=<OCB=90^0$.

Const. : Join OA and OB. (But why ? There is no answer and no justification for this step at this stage. The necessity will automatically get justified after you have proved the proposition with the help of this construction).

Proof : In $\triangle$s OCA and OCB. (But where is the necessity of referring to these two triangles ? Why should this and only this step strike the learner ?)

OA=OB. (radii of the same circle.)
OC=OC. (common.)
AC=BC. (given.)

$\therefore \triangle OCA=\triangle OCB$. But what is the reason that you should prove the congruency of these triangles ? Why not do something else ?)

Hence $<OCA=<OCB=90^0$.

Q.E.D.

Ex. 2. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{ac-2b^2}{b}=\frac{c^2-2bd}{d}$

Synthetic proof :—

$\frac{a}{b}=\frac{c}{d}$ (It is known, and hence the starting point.)

Subtract $\frac{2b}{c}$ on both sides ? (But why ? Why and how should the child remember to subtract $\frac{2b}{c}$ and not any other quantity ?)

$$\frac{a}{b}-\frac{2b}{c}=\frac{c}{d}-\frac{2b}{c}$$

or $$\frac{ac-2b^2}{bc}=\frac{c^2-2bd}{cd}$$

or $\frac{ac-2b^2}{b}=\frac{bd^2-c^2}{d}$ $\left(\text{cancelling } \frac{1}{c} \text{ on both sides}\right)$

Hence the identity is proved.

Ex. 3. In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing the acute angle minus twice the rectangle contained by one of these sides and the projection upon it of the other.

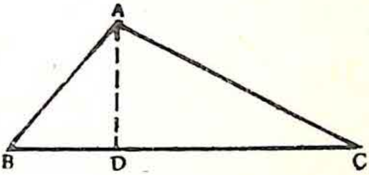


Fig. 5

Given : An acute <d. △ ABC, acute <d at B. BD is projection of AB on BC.

To prove : $AC^2=AB^2+BC^2-2BC.\ BD$.

Proof : In rt. <d △ ACD

$AC^2=AD^2+CD^2$ (But why should this be the first step of 'Proof' ?)

Expanding the R.H.S. further.

$$\begin{aligned} AC^2 &= AD^2+CD^2 \\ &= AB^2-BD^2+(BC-BD)^2 \qquad [\because CD=BC-BD] \\ &= AB^2-BD^2+BC^2+BD^2-2BC.\ BD \\ &= AB^2+BC^2-2BC.\ BD. \qquad \text{Q.E.D.} \end{aligned}$$

[The fundamental reason behind each and every step has not been given.]

Merits

1. It is a short and elegant method.
2. It glorifies memory.

Drawbacks

1. It leaves many doubts in the mind of the learner; and offers no explanation for them.
2. Without a satisfactory answer to so many questions that arise in synthesis, the pupil is perplexed when a new problem is set to him. Even the recall of all the steps of synthesis may not be possible for the learner.
3. It does not provide full understanding.
4. There is no scope of discovery and thinking in this method.
5. Memory work and home work are likely to become heavy.

CONCLUSION

The pupil taught by the method of synthesis is just like a man led blind-fold to the desired goal. Since analysis is a lengthy method, it needs the help of Synthesis for the removal of this defect. It will not be useful if it is not followed by synthesis. Synthesis is the complement of analysis; and, in the teaching of mathematics, the two should always go together. Analysis leads to synthesis, and synthesis makes the purpose of analysis clear and complete. In the teaching of mathematics, analysis is done for synthesis-analyse in order to synthesise, i.e., facts or things are separated in order to find out how these can be knit together to form one whole. Analysis helps in understanding and synthesis helps in retaining knowledge. Analysis forms the beginning and synsthesis forms the follow-up work. The two are interdependent. Teacher should realise that he may offer help for the analytic form of the solution and that synthetic work should be left to the pupils.

A comparative study of the two methods should be of interest to the reader.

Analytic Method	*Synthetic Method*
1. It proceeds from the unknown to the known facts.	1. it proceeds from the known to the unknown facts.
2. It starts from the conclusion and goes to the hypothesis.	2. It starts with the hypothesis and ends with the conclusion.
3. It is a process of thinking (exploration.)	3. It is a product of thought.
4. It is a process of exploration and demands thought.	4. It is a process of presentation of the previously discovered facts.
5. It pulls apart or analyses the statement under solution.	5. It puts together or synthesises known facts.
6. It is a general method.	6. It is a special device.
7. It is lengthy, awkward, slow, roundabout, and involves trial and error.	7. It is concise, elegant, quick, straightforward, and does without trial and error.
8. It answers satisfactorily any question that may arise in the mind of an intelligent pupil.	8. It does not satisfy the doubts and questions arising in the mind of the learner.
9. It is a method for the thinker and discoverer.	9. It is a method for the crammer.

10.	There are close contacts between the teacher and the taught.	10.	There are no such intimate contacts between them.
11.	The students can recall and reconstruct easily any steps if forgotten.	11.	It is not easy to recall or reconstruct any forgotten steps.
12.	It develops originality.	12.	It develops memory.
13.	It is informal.	13.	It is formal.
14.	It is psychological.	14.	It is logical.
15.	It is formational.	15.	It is informational.
16.	It is based on heuristic lines.	16.	There is no heuristic approach in it.
17.	It is the fore-runner of synthesis.	17.	It is the follower of analysis.

LABORATORY METHOD

Mathematics is a subject which has to be learnt by doing rather than by reading. The doing of Mathematics, gives rise to the need of a suitable method and a suitable place. Laboratory method and mathematical laboratory are the proper answer to it. This activity method leads the pupil to discover mathematical facts. It is based on the principles of 'learning by doing', 'learning by observation' and proceeding from concrete to abstract. In one sense, it is only an extension of the Inductive Method. It is a more elaborated and practical form of the Inductive Method. Pupils do not only listen for information, but do something practically also. Principles have to be discovered, generalised and established. The method, if properly used, should help in the removal of the abstract nature of mathematics. It makes the subject interesting as it combines play and activity.

PROCEDURE

The construction work in geometry is on the whole a laboratory work. The drawing of a line, construction of an angle, construction of a triangle or a quadrilateral or a parallelogram, etc., all involve the use of some equipment, and therefore their nature is that of practical or laboratory work. There can be many more illustrations to explain the procedure. For calculating the area of a triangle, cut out a triangle of cardboard. Find the weight of a unit area of the cardboard and then find the weight of the triangle. This weight divided by the weight of the unit area of the same cardboard will give you the area of the triangle.

Similarly, proving the congruency of triangles by the method of placing is basically laboratory work. Calculation of the volume of a solid (cuboid or cylinder) can be done with the help of a graduated cylinder which is filled with water to a certain height. When we immerse the solid in this water, the rise in water will give the volume of the solid. After doing so practically with a number of similar solids, the formula can be established.

To prove the indentities connected with $(a+b)^2$, $(a-b)^2$, a^2-b^2, etc., the help of cardboard models can be taken. The models will preferably be prepared by the students for themselves. The proof that, the sum of angles of a triangle is equal to two right angles. can be attempted practically also at the initial stage. Take triangles of different sizes, ask the students to sum up their angles after measuring them. The sum in all cases will be the same. Every geometrical proposition may be proved practically before it is attempted theoretically. Every process like counting, notation, addition, subtraction, multiplication, division, weighing, measuring, equalising, etc., of course with the help of some relevant material, is a kind of practical laboratory work. Especially in lower classes, the introduction of this work is more essential, desirable and practicable.

Merits

1. It is interesting and joyful for the learner. He likes to do something with his own hands.
2. It is based on two sound psychological principles, namely, "Proceed from concrete to abstract" and "Learn by doing".
3. The learner acquires a clear understanding of the subject. He finds or discovers facts with his own effort.
4. It provides great scope for independent work and individual development. It helps in the growth of self-reliance.
5. It inculcates the spirit of cooperation and exchange of ideas when the students are required to perform laboratory work in groups.
6. A successful experiment is a source of joy and encouragement to the learner.
7. Shyness of hands is removed, as the learner has to handle apparatus and material.
8. The application of mathematics becomes increasingly evident to the learner. Thus the subject becomes functional and meaningful to him.
9. Some topics of mathematics are best understood through this method.

Drawbacks of Laboratory Method

1. The Method is very expensive. Every school cannot afford to spend a large amount of money on laboratory equipment.
2. Laboratory work is not typical mathematical work; and hence this method, if exclusively used, does not give any training to the learner in true mathematical thinking.
3. It acquaints the students with facts and not with mathematical reasoning.
4. All the topics of mathematics cannot exclusively be taught by this method.
5. It needs thorough planning and supervision, otherwise students may just play with instruments without deriving any substantial gain. Since teacher will be required to pay individual attention, it may not be practicable in large classes.
6. It is an exceedingly laborious and slow method.
7. It is not at all easy to make the students discover mathematical facts experimentally, especially in lower classes.
8. It may sometimes degenerate into a kind of manual training only.
9. The tendency of cooking up results or copying may develop among them, as it is not easy to check.

CONCLUSION

It is a difficult and lengthy method, but can prove exceedingly profitable if properly employed. A lot depends on facilities available, which vary according to what a particular school can offer by way of staff, laboratory accommodation and equipment. Cost can be reduced a bit, if equipment is improvised in the school itself. Young children will then be fascinated by this method. This method should be "a must", where circumstances favour.

THE PROJECT METHOD

It is based on John Dewey's philosophy of pragmatism. According to Dr Kilpatrick, "A project is a unit of whole-hearted purposeful activity, carried on preferably, in its natural setting." Stevenson defined it as, "A problematic act carried to its completion in its natural setting." "A project is a bit of real life that has been imported into the school", thus says Ballard about it. Project plan is a modified form of an old method called, "concentration-of-studies"

The main feature of the 'concentration-of-studies-plan' is that some subject is taken as the core or centre and all other school subjects, as they arise, are studied in connection with it. It is based on the principle of learning by doing. It assumes that knowledge grows by application. Moreover, it is based on the fact that the different branches of knowledge are not separable, though they are studied separately for some superficial convenience. Knowledge is indivisible, and project method is a method in accordance with this natural correlation. It is a method of spontaneous and incidental teaching. As the project progresses, the learner or group of learners goes on picking up any piece of knowledge that may happen to be relevant, necessary and useful.

PROCEDURE

There are two types of projects :

(a) individual project, carried out by a single individual; (b) the social project which is carried out by a group of individuals.

To complete any project, we have five stages in actual practice.

(i) Providing a situation.
(ii) Choosing and purposing.
(iii) Planning of the project.
(iv) Executing the project.
(v) Judging the project.

There should not be any rigidity about these stages. Modification may be made according to the nature of the project and the mental level of students. The teacher has to give the pupils an equitable distribution of work, according to their abilities and stamina. The whole project work should be reviewed, judged and evaluated in the end.

To begin with, a project should arise out of a need felt by pupils. It should not be forced on them. It should be purposeful and significant. It should look important, as its importance will enhance the value of work. It must be interesting. The absorbing interest will sustain pleasure.

Ex. 1. 'Understanding about a local factory' may be a project for the students. The following different aspects may have to be dealt with for the accomplishment of the project.

1. The name and location of the factory.
2. The nature of its raw material and its produce.
3. The source of its raw material.
4. The geography of its raw material.
5. Whether it is an enterprise of an individual or of share-holders. In the later case, what are the shares of the different share-holders ?

6. The number of persons employed in the factory.
7. Their ranks, grades and salaries.
8. How much per cent of an individual's salary is deposited in provident fund ?
9. The annual income of the factory.
10. The share of profit for different share-holders.
11. The progress of factory in the matter of production and income from year to year. The graphs of this progress.
12. The scope of expansion of the factory.
13. The relation of the factory with other local social institutions.
14. What type of qualifications will enable a person to get employment in the factory ?
15. An essay on the visit to the factory.

While answering these and many other possible queries, the students will acquire a thorough knowledge of the local industry and at the same time learn many relevant topics of different subjects. The teaching of mathematics will also come in incidentally.

Ex. 2. 'Celebration of Republic Day' may be another project. The project can have the following different aspects :

1. Why do we celebrate this day ?
2. The school programme of its celebration.
3. Estimated expenditure for different items; budget preparation.
4. From where to arrange for the expenses ?
5. The account of collections from the locality, if any.
6. The account of collections from amongst the students, if any.
7. The preparation and organisation of different items of the programme.
8. Making purchases for the programme.
9. Keeping accounts of the expenses incurred.
10. Execution of different programmes.
11. Preparing the report of the programme.
12. Evaluation of the entire celebration.

These are the directions for the students to carry out the project. An enthusiastic planned celebration will necessitate the learning of many valuable ideas concerned with different subjects. As soon as the students take up an aspect of the project, the teacher will utilise the opportunity to enable the students to learn something.

MERITS OF THE PROJECT METHOD

1. It is based on psychological laws of learning. Education is related to child's life and is acquired through meaningful activity.
2. It upholds the dignity of labour.
3. It introduces democracy in education, because it necessitates cooperation among students and their acting together for a common cause.
4. It brings about concentration of studies and correlation of activity and subjects.
5. It emphasises problem-solving rather than cramming or memorising.
6. It inculcates social discipline through joint activities.
7. It develops self-confidence and self-discipline.
8. A project tends to illustrate the real nature of a subject and produce a spirit of enquiry.
9. Projects can be used to arouse interest, justify the study of topics, encourage initiative and give the students joy at the successful completion of a given work.
10. Teaching becomes incidental because the child expresses willingness and is goaded by the desire to learn.
11. It challenges the capacities and abilities of the child and puts him on the track to think and act.
12. There is an opportunity for mutual exchange of ideas.

DRAWBACKS OF THE PROJECT METHOD

1. Mathematics cannot be taught purely by this method. Incidental teaching will not suffice; planned teaching will have to be added to it.
2. The student will not acquire skill and efficiency without collective and individual drill which will be possible through planned teaching only.
3. There is no saving of time, energy and effort.
4. A single opportunity for practical experience cannot develop or promote computational ability.
5. Systematic and continuous teaching is not possible.

CONCLUSION

This method brings life to the school atmosphere. Learning becomes a cooperative affair. Its approach is scientific and psychological. As it is not suitable for drill and continuous and systematic teaching, it is not very desirable to use it freely. If the teacher can devise and plan a good project on something, the students will gain a lot. It leads to understanding and develops the ability to apply knowledge. Its occasional use will always remain an effective tool in the hands of the teacher. The teacher has to work as a careful guide during the execution of the project. Costly projects should be avoided. Unsupervised and unevaluated project work will also be of no use.

SOME SUITABLE PROJECTS

A few good and suitable projects are listed here for the teacher's guidance :

1. A picnic—its organisation, expenditure, etc.
2. Purchase of craft material.
3. Running the hostel mess.
4. Model of Bhakra Dam.
5. Model of the Panjab State.
6. The sports-day.
7. A Variety Programme.
8. Model of the village.
9. Running a cooperative shop in the school.
10. School Bank.
11. Collecting rates of a few commodities from a number of shops in the bazar.
12. Uses of mathematics in large business.
13. Collecting data about national and provincial budgets.
14. Bus and Railway fares from their town to important stations.
15. Mathematics in the classroom.
16. Mathematics in the kitchen.
17. Graphs of employment, death rate, population, etc., in the locality.
18. Mathematics in a post-office.

TOPICAL METHOD

It is the opposite of Concentric Method. Concentric Method involves the breaking up of a topic into suitable portions, whereas Topical Method aims at keeping it intact. A topic is taken as a unified whole or as an unbreakable unit. It is based on the principle that any topic when begun should not be left half done. It should be finished in its entirety, before the next topic is taken.

PROCEDURE

'Topical Method' is more a system of arrangement of subject-matter than a method of teaching. Its adoption depends on a suitable organisation of the syllabus. The topic is to be taught at a stretch, without a break or a gap. The other approach to this method is that a topic is selected and is made the basis of many other topics. The selected topic becomes the centre of correlation. While dealing with the unitary method, the students can be acquainted with time and work, simple interest, average percentage and even, to some extent, with simple equations of algebra. Topics of other subjects may also be mentioned.

MERITS OF TOPICAL METHOD

I. Continuous teaching of a topic not only will save the students from divided attention, but may ensure their full and whole-hearted concentration on the topic in hand also. A natural link and sequence will exist in the day-to-day work in the classroom. The student's complete attention, ability and capacity will be directed exclusively to the topic under study for a sufficiently long time.

II. When a topic is treated as centre for other topics, it facilitates the learning process. It illustrates the advantages of correlation.

Its Drawbacks

I. Keeping psychological reasons in view, it will be foolish to take up a topic like Area in the 4th class and try to finish it at a stretch. The students may be capable of understanding the elementary portion of the topic at that stage, but will certainly not be able to attempt the most difficult problems at that very stage.

II. It is but natural that the interest of students in any topic will cease or wane after a month or so of its start.

III. The method does not provide any opportunity for year to year revision. Most of the knowledge acquired in earlier year is likely to be forgotton by the time the students leave the school. At the school leaving stage the students are wanted to do a final and overall revision so that when they go out everything should be fresh in their minds. But topical method does not provide for such revision.

CONCLUSION

The method or system does not possess any significant advantage. Moreover, there is a better substitute in the form of Concentric Method. The only advantage of this method is that a topic is made a centre for other topics, which can prove a good step towards correlation. As a method, it is only an idea, which should be kept in mind to be used only at the appropriate opportunity.

CONCENTRIC METHOD

The method implies widening of knowledge just as concentric circles go on extending and widening. It is more appropriate to call it a system of arrangement of the subject-matter. The study of a topic is spread over a number of years. Beginning from a nucleus, the circles of knowledge go on widening year after year. An elementary knowledge is given in the introductory year, something more is taught in the next year, something still more is added in the subsequent year and so on. A steady progress is maintained in the acquisition of knowledge of a topic for a sufficient number of years continuously. .

PROCEDURE

A topic is broken into a number of portions and the portions are allotted to various classes. The criteria are difficulty of portion and power of comprehension of students in an age group. Any gaps of knowledge are filled up year after year. Full-fledged knowledge of the topic is ultimately provided. It is mostly concerned with year to year teaching, but its influence can be exercised in day-to-day teaching as well. Knowledge to be given today should follow from the knowledge given on previous days and should lead to the teaching on the following days.

ITS MERITS

I. It is a suitable system of arrangement, because it is highly unpsychological to take up a topic like "Time and work" with a particular class and try to deal with all its aspects in the same class.

II. It enables the teacher to do a portion of a topic according to the receptivity of learner. The learners get as much as they can grasp.

III. Continuance of the study of a topic for a number of years makes its impression more lasting. Previous knowledge or comprehension a topic is revised year after year, which helps in its retention.

IV. It does not allow the teaching to become dull. The interest of the student remains alive for the short duration of the study of the portion of a topic. Every year, a new interest can be given to the topic.

Its Drawbacks

1. If a portion is too long, the interest is likely to flag; and if a portion is too short, it is not likely to make any lasting or permanent impression on the learner's mind.
2. It may prove harmful, if a teacher becomes over-ambitious and exhausts all the possible interesting illustrations in the introductory year. Those who have to teach the topic in the subsequent years are left with nothing to create interest.

CONCLUSION

The system is worthy of being adopted in the arrangement of subject matter. The organisers have to be careful so that a portion is neither too long, nor too short. It should be a well-measured portion. To remove its other drawbacks, it is proposed that the teaching of same students should remain with the same teacher throughout their schooling. Then the teacher will always reserve some illustrative examples for each year, so that each new portion of the topic can be started with fresh interest and vigour. It should be possible to introduce the topic with interesting illustrations every year.

PROBLEM METHOD

It can also be called problem-solving method. Here problem itself is the crux of the problem. It pre-supposes the existence of a problem in the teaching-learning situation. A problem is a sort of obstruction or difficulty which has to be overcome to reach the goal. According to Yoakam and Simpson "a problem occurs in a situation in which a felt-difficulty to act is realised. It is a difficulty that is clearly present and recognised by the thinker. It may be a purely mental difficulty or it may be physical and involve the manipulation of data. The distinguishing thing about a problem however is that it impresses the individual who meets it as needing a solution. He recognises it as a challenge . . ."

Life is an arena of problems. L.A. Averill has said, "The only worthwhile life is a life which contains its problems; to live without any longings and ambitions is to live only half-way." A human child has to meet and solve problems as he grows—problems which present themselves in his physical surroundings, his intellectual associations and in his social contacts. These problems grow in number and complexity as he grows older and older. His success in life is in large measure determined by the individual's capacity and competence to solve them. Problems exist for him at every step; his growth, development and living lies in their solution. In school, the child is to be trained in the art and craft of problem-solving.

The problem method aims at presenting the knowledge to be learnt in the form of a problem. It begins with a problematic situation and consists of continuous, meaningful, well-integrated activity. The problems are set to the students in a natural way and it is ensured that the students are genuinely interested to solve them.

Mathematics is a subject of problems. Its teaching and learning demands solving of innumerable problems. Efficiency and ability in solving problems is a guarantee for success in learning this subject.

PROCEDURE

The procedure of problem method is almost like that of the project method. It can also take the form of inductive-deductive methods. The procedure involves the following steps :

1. Recognising the problem or sensing the problem.
2. Interpreting, defining and delimiting the problem.
3. Gathering data in a systematic manner.
4. Organising and evaluating the data.
5. Formulating tentative solutions.
6. Arriving at the true or correct solution.
7. Verifying the results.

It is a research-like method. It involves scientific thinking as a process of learning. Its relationship with inductive-deductive method is very intimate.

According to exclusive inductive approach the method will take the form of the steps—sensing the problem, analysing the situation, organising information, framing solutions, elimination and verification.

Whereas according to exclusive deductive approach the steps will be—understanding the problem, collecting information, reviewing, drawing inference and verification.

But how to employ this method in the teaching of mathematics? Take for example that finding the volume of a cylinder is a problem before the class. Its formula has to be developed on the basis of the earlier formula for the volume of a cuboid. While analysing the problem, it gets connected with the previous knowledge that volume of any regular solid can be found by multiplying area of its base with the height of the object. The given information is so organised that it becomes the required information. The area of the base of the cylinder is found by an already known formula and method. Then the required formula is obtained by multiplying this area with the given height. For the purpose of verification it is applied to a number of similar problems or situations and the results are checked. The solutions to the problems always come from the students. The teacher remains in the background and directs or guides the student activity from that position.

ITS MERITS

1. Problem-solving in schools prepares the pupils to solve the problems of life. It approximates to life. Facing and solving problems is the true nature of life itself.
2. The method involves reflective thinking. Therefore it stimulates thinking, reasoning and critical judgment in the students.
3. The pupils learn by problem-solving which is the method of learning by self effort.
4. It develops qualities of initiative and self-dependence in the students as they are to face the problematic situations themselves.
5. It is a stimulating method. The problem is a challenge. Once it is properly recognised, it acts as a great motivating force and directs the students' attention and activity.
6. It is especially suitable for mathematics which is a subject of problems.
7. In it there is strong motivation, tension and mental activity which are the conditions for effective learning.
8. It serves individual differences. There are no limits on student achievement. He can solve any number of problems in a specified time and make progress accordingly.

9. It develops desirable study habits in the students. They get engaged in analysis of the problem, reflective thinking, systematic data gathering, verification and critical study.
10. It is a method of experience-based learning. Problem-solving by self-effort is an experience of its own type. Such an experience is found missing in the lectures and reading from the textbooks.
11. There is possibility of close contact between the teacher and the taught. Every student needs individual guidance from the teacher. The teacher comes to know the difficulties which the students face and helps them accordingly.
12. The students get valuable social experience like patience, cooperation, self-confidence, etc.

Its Limitations

1. Its limitations are largely due to its ineffective use. There are, otherwise, no limitations inherent in it. "If the teacher is not able to think reflectively, does not have an attitude of critical enquiry; or when the classroom situation is dominated by him, and the atmosphere is that of recitation and of ready-made answers, the problem-solving method is going to fail."
2. It is difficult to organise the contents according to the requirements of this method. It is difficult to frame really good problems and to introduce them at every step.
3. It is a time-consuming method. The progress of the students is bound to be slow.
4. All the topics and subject areas cannot be covered by this method.
5. The method does not suit the students of lower classes. They do not possess enough background for scientific approach to problems.
6. When the structure of the problem itself is not upto the mark, or when the problems chosen are unreal and artificial, the method will not be applicable.
7. Textbooks written in the traditional style do not help in the use of this method. There is absence of suitable books for reference and guidance.
8. Teacher's burden becomes heavier. Real scientific approach to problems necessitates a lot of study and preparation on his part.

9. It is an intellectual approach in learning. Mental activity dominates in this method and there will be neglect of physical and practical experiences.
10. In case the assigning of problems or proposing of problems becomes the teacher's main job, then the procedure obviously smacks of authority, spoon-feeding and artificiality.

CONCLUSION

On the whole, problem-solving is a suitable approach in the teaching of mathematics. It develops in the learners the ability to recognise, analyse, solve and reflect upon the problematic difficulties. This method is akin to other progressive methods like heuristic, project or inductive. At the same time it has its own distinct entity, its own procedure and its own way of doing things. There is a wrong tendency not to consider problem method as a distinct procedure, but to use it every where in every form. Problem-solving helps us at every step in our teaching-learning process. The only precaution about its use is to select the problems which have definite educational values and to set-up an intellectual atmosphere in the class for proper problem-solving.

AN ASSESSMENT

A number of methods of teaching have been discussed. Some of them have been recommended for use, some have been disapproved and some have been recommended for use with caution. Out of all the available methods, every teacher has to make his own choice. This choice cannot to be a whimsical choice, but it will have to be made in a rational way keeping in view the facilities available and the nature of work to be done. Certain methods have been fully approved whereas others have been partially approved. There are still others which have been outrightly disapproved. After understanding all of them, the teacher has to make his own decision as to which of the methods is best for him. Each of the methods like inductive-deductive, analytic-synthetic, problem-solving, and heuristic can be reasonally supported to be one of the most suitable methods. It is in no way difficult to argue in favour of either of these. But supporting any one of the methods after ignoring all others may not be a very sound proposition. Our preference for any one of them alone may deprive us of the possible benefits from all others. There are bound to be available some good qualities even from the worst of the methods. It may not be a sound practice and a good bargain to select one of the good methods as the best method and forget all other methods for ever. Perhaps, even the worst of the methods has also got to show us the way in some special circumstances. No teacher can afford to rely for ever and in all circumstances only on one method, however good that may be. Even the best of the methods may not succeed in certain conditions or in the case of certain topics. It is wrong for a teacher to become slave of only one method, and think that the same method will serve his purpose for ever. He should not cultivate this notion that all other methods excepting one are no good and therefore do not concern him. It will not be educationally sound on his part to commit himself to one method and to severe connections with all others. He should try to imbibe the good qualities of all the methods and should so improve his command over all of them that he can even make the best out of the worst. He will keep his knowledge of all the methods up-to-date and will exploit their qualities to the maximum while reducing their shortcomings to the minimum. He will disown only the defects of a bad method and will try to have the advantage of anything found appreciable in it.

The teacher will keep himself on the right side of every method. Broadly speaking he will be a master of every method in the real sense. With the passage of time, he will evolve his own method and will formulate a teaching style of his own. His best method will be his own individualised and personalised method, which is the result of his rich and varied experience in teaching. His method, an amalgam of all the known qualities, will obviously carry the stamp of his individuality. He will teach best as he learns and experiences teaching.

Preferably he will adopt heuristic approach as an overall procedure in teaching. His method will be a pupil-dominated method rather than a teacher-dominated method. He will give maximum opportunity of participation to the students and will impose himself the least on the classroom work. Whatever method he adopts, the heuristic approach should always be made to prevail. Let it be the student's concern and privilege to discover things for himself, and let not the teacher tell him everything from his side. Spoon-feeding has to be completely avoided. The heuristic approach will be the mainstay of any teacher of mathematics. However, his method will be a happy combination of all the methods minus the drawbacks of all of them. He will be watchful of the defects of even the best of the methods and fully employ the good points of even the worst of the methods.

The twin combinations of inductive-deductive and analytic-synthetic methods are recommended as his day-to-day methods. The inductive-deductive combination will be more suitable in the case of arithmatic and algebra whereas analytic-synthetic will find greater application in plane geometry, trignometry and solid geometry.

The teacher need not stick to the same method always. Even the best of the methods will become monotonous with continuous use.

His method must carry in it some variety and newness in order to sustain interest. No doubt he will stick largely to his favourite method. but he should occasionally introduce a new method in his method. When he finds that the students are showing lack of interest towards the best method being adopted by him, he should bring in a different method just for the sake of change.

In some topics, it will be quite interesting and rewarding to use project method or laboratory method. These methods may be of great help to break the monotony created by theoretical methods. Activity involved in these methods will provide a good change. In addition, these methods possess some distinctive qualities and inculcate good characteristics in the learners, which no other method can impart equally well.

Undoubtedly, in some cases, it will be not only obligatory but suitable to use dogmatic or lecture approach. May be, they are considered to be the most unsuitable methods in this age of child-centred education; but still they have some place in the total programme of teaching. It is wrong to ban them completely. There are many descriptive topics like budgeting, insurance, taxation, etc., where a good lecture may create a real interest. Similarly a dogmatic approach may suit on occasions where a number of things have to be remembered thoroughly and accurately. It is in the interest of the students if the teacher becomes dogmatic to demand complete exactness from them.

The problem method is also reasonably suitable as an approach towards problem-solving in mathematics. Let there be a general attitude to accept every mathematical problem of a difficulty as a new challenge. The same should be met with pleasure and confidence. The solving of problems in mathematics should not be mechanical or routine-like affair, it should rather be an experience of scientific or reflective thinking. In the composite method of teaching to be adopted by an intelligent teacher of mathematics, problem method will claim its well-deserved place.

There are some less important methods like topical and spiral. They will also not be completely ousted from the day-to-day programme of class teaching. A teacher who understands them can certainly take advantage out of them whenever there arises an opportunity.

In the end, we can safely conclude that it is wrong to name one single method as the best method. A good teacher will so digest or absorb them all that he evolves his own method comprising good points of all the methods. He will not permit any of the methods to become his master but will remain a true master of all of them.

EXERCISES

1. Which is the best method of teaching mathematics according to your judgment? Support your preference with arguments.

2. A practical teacher of mathematics cannot employ exclusively any of the methods of teaching. His method, in all probability, will be the combination of almost all the methods. Discuss.

3. Illustrate and discuss the inductive-deductive methods of teaching. What is the justification for their combination?

4. Explain the use of analytic-synthetic methods of teaching. Compare them for their similarities dissimilarities.

5. Which is the teaching method most popular with the present day teachers of mathematics? What are its major drawbacks and how can improvement be brought about under the existing conditions and limitations?

6. Heuristic spirit is the golden rule which a teacher of mathematics should never forget, whatever method of teaching he may adopt? Elaborate and discuss.

7. How will you employ the project method for the teaching of mathematics? How can you employ it in actual practice?

8. What is the place of laboratory method in the teaching of mathematics? How can you employ it in actual practice?

9. What do you understand by problem method. What is its scope in the teaching of mathematics? How will you employ it as your day-to-day method?

10. Write short notes on:
    - (i) Unsuitability of lecture method.
    - (ii) Dogmatic method.
    - (iii) Concentric method.
    - (iv) Need of a standardised method for the teaching of mathematics.
    - (v) Limitations of heuristic method.
    - (vi) Combination of analytic-synthetic methods.
    - (vii) Procedure of project method.
    - (viii) Drawbacks of problem method.
    - (ix) Merits of laboratory method.

CHAPTER EIGHT

AIDS TO TEACHING AND MATHEMATICS LABORATORY

THE USE OF sensory aids in the teaching of mathematics is of recent origin. In fact, all teaching has always involved the communication of ideas through the senses either orally through the medium of speech, or visually by the use of written or printed material. Textbooks, writing materials, geometrical instruments and the blackboard (all these are sensory aids) have long been regarded as indispensable equipment for mathematics classes. For many years resourceful teachers have used models, instruments, drawings, and other devices to stimulate interest and facilitate learning. But for a long time the potential values of these supplementary devices were fully realised only by exceptional teachers. Only lately has there been a concerted effort among leaders in mathematics teaching at making teachers alive to these possibilities.

Mathematics is essentially a subject, where doing is more prominent than reading. That is why a certain amount of equipment is indispensable in order to make even a start in this subject. Moreover, it is held by a vast majority of people that mathematics is a dry and difficult subject, full of abstract things. The result is that students take very little interest in it. To create the necessary interest is a constant problem for the teacher. This subject demands the use of aids at every step.

Equipment for mathematical instruction falls into two categories: (1) that which the student needs in order to pursue his own individual study, and (2) that which can be used in common and has to be provided by the school in the Mathematics Laboratory.

The former category includes such obvious necessities as textbook, writing equipment, simple drawing and measuring instruments in the form of a geometry box, and in some cases special equipment such as the slide rule.

PERSONAL EQUIPMENT OF THE STUDENT

As mentioned above this equipment includes instruments which the child has to frequently use for his individual work. They constitute the minimum essential equipment—equipment without which any effective work in mathematics is impossible. At the same time, this equipment is not extensive; rather, it is simple and inexpensive. It is very important that each student should have his own personal equipment. He should provide himself with textbook, notebook, pencil, rubber, pen and a geometry box. Borrowing these things is a lazy habit, which wastes the time of both the borrower and the lender. This habit should be discouraged straightaway. The instruments which he must have should, of course, be simple, substantial and inexpensive.

EQUIPMENT OF THE MATHEMATICS LABORATORY

In addition to the personal equipment of the child there are numerous special items of equipment which can be provided advantageously for general use in the classroom or in the mathematics laboratory. The feeling is rapidly growing that elementary mathematics can profit greatly by the introduction of field and laboratory work. This will in addition popularise the subject. The mathematics laboratory is no longer a misnomer : it has become in many places an established medium of giving meaning and interest to the subject.

Mathematical equipment should be made available through mathematics laboratory as far as possible. Expenses on such equipment should form as legitimate a part of the school budget as expenses on equipment and supplies for athletics, sciences, fine and industrial arts and crafts. School authorities and administrators should give up the wrong attitude that since mathematics is not listed among the laboratory subjects, it requires no special laboratory equipment. This point of view is not in line with the modern educational thought, nor is it true to the practice of progressive schools.

PLACEMENT OF THE MATERIAL

Some place for storage is a prerequisite of mathematics laboratory. There must be some almirahs and a room to store and stock the articles before any purchase of the things is made. The place may be provided by having an almirah in every classroom or by having a few almirahs in one room. Without a place, the material will lose half its age. The things will remain scattered, unarranged, exposed to dust, etc. They are likely to lose their colour and attraction.

Now follows an account of the different items of equipment and their possible advantage as aids to teaching.

Blackboard

This is the first and foremost of all the items of mathematical equipment. It must be there even if anything else is not there. This is the minimum equipment. But sometimes it need not even be included in the list of special equipment, as it is a taken for granted part of every classroom. Blackboard may verily be called the second tongue of the mathematics teacher. But it is a pity that many teachers fail to use it. As soon as the teaching of this subject starts, the use of blackboard should begin and go right upto the highest education in it.

The mathematics teacher should be good at blackboard writing and drawing. He has to write on it statements of problems, their solutions, some points needing special emphasis, and has to draw on it the relevant diagrams. It should not be used in the manner that the teacher goes on writing and drawing and the students go on copying. The entire matter to be written on the blackboard should be developed with the active cooperation of the students. If possible, the students should also be given the opportunity to write and draw on the blackboard.

A roller blackboard with four sheets will be most suitable from the point of view of writing space and convenience. One of these sheets may be a graphed one. Chalks of different colours may be used to make the writing and drawing colourful and attractive. Effective use of blackboard necessitates possession of blackboard instruments.

Most aspects of mathematics can be clarified only through writing. Verbal explanation will not suffice in such cases.

Concrete Material

This type of material should include many concrete objects, such as beads, ball frames, number cards, sticks, coloured balls of solids, even pebbles, seeds, toy money, weights, other measures, etc. This list can be further extended by including specially manufactured things.

Teach a number Kit

It consists of a set of wooden blocks each $(1\frac{1}{2}'' \times 1\frac{1}{4}'')$. In it, a different coloured block is used for each digit from one to ten. Each block is labelled with the numeral and the word it represents. The blocks vary in thickness to facilitate comparison and recognition.

Place-Value pockets

It is a wooden box divided into three or four portions labelled as

"Thousands", "Hundreds", "Tens" and "Units" respectively. It can be used to demonstrate place value, to illustrate carrying in addition and multiplication, and to illustrate borrowing in subtraction.

FRACTIONAL PARTS

It consists of disks which are dissected respectively into halves, fourths, eighths, thirds, sixths, fifths etc. It provides materials necessary for a meaningful study of fractions.

Even a magnetic needle with a bar magnet will not be out of place in explaining the concept of locus.

The following articles may also be included in this category:— clock faces, cubes, spheres and cylinders etc. The laboratory should also stock cardboard, chartpaper, colours, inks, cutting material, simple work appliances, measuring tapes, yard sticks etc.

CHARTS

The charts help in creating a suitable subject atmosphere in the classroom, and in elucidating various points.

These charts can cover a vast range of mathematical topics, such as coins; weights and measures: prices of different articles; school and class pass percentage; different geometrical figures and their qualities; different kinds of angles; different kinds of geometrical figures; different kinds of triangles; different kinds of quadrilaterals: different kinds of polygons; vocabulary of a triangle, a quadrilateral or a circle; different methods of purchase; different priced articles of a post office; income and expenditure of a person or his budget; different types of bank drafts or cheques; arithmetical terminology such as fraction, average, area, profit and loss, interest, ratio, etc; average daily attendance; average temperature in a week or in a month; average expenditure of a family; life calenders of famous persons and their portraits.

These charts can also pertain to the proofs of almost all the propositions and their exercises and corrollaries.

Charts should be preferably multicoloured and artistic. The beauty and colourfulness will certainly enhance their appeal.

Charts help save the time of the teacher, because instead of drawing them himself on the blackboard, he can depend upon the pre-drawn diagrams. Moreover, it is not always possible to draw a diagram on the blackboard with fair accuracy.

Some of the good charts should remain hanging on the walls of the classroom. It is better if the charts of the topic, being then dealt with, are hung more prominently and in a fairly large number. Charts should be changed with the topic. When a new topic is started, its relevant charts should replace the previous ones; and, as soon as the topic is over, its charts should make room for the charts on the next topic.

Some commercially prepared charts are also available; such as the international metric system charts; a time chart; linear measure chart; change for a dollar chart; charts on kinds of subtraction; charts on temperature, weights, decimals, adding fractions, long division, etc. But locally prepared charts are always preferable, whether they have been prepared by the teacher or by the students, provided they are adequate in quality and quantity.

MODELS

Models are very useful as an aid, because they can be handled and manipulated. Their main merit is that they afford pleasure in their making. This aid has a creative value for anyone who may make them.

Even a square piece of cardboard is a model. All other geometric forms can be cut out of cardboard or a thick paper and used as models.

More illustrations of models are given below :

(i) Take a circular piece of thick paper to explain the angles at a point, and their relation with the whole circumference of the circle. Half of the circumference subtends an angle of 180^0 at the centre, one fourth circumference subtends an angle of 90^0 at the centre, and one sixth of the circumference subtends an angle of 60^0. While constructing angles in geometry, one has to depend on this principle, but one has not to attach so much importance to the explanation of the principle when the aid is used.

(ii) There is a device known as a hundred board, which contains a hundred chips or disks. When you take out one chip, that means one from a group of hundred or one per cent. With a little practice on this type of device, the students can easily understand the value of the fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{10}$ and so on in terms of per cent.

(iii) A direct way of proving the sum of the angles of a triangle is by folding. Cut a triangle ABC out of paper. Fold it as shown in the diagram. The outer figure is the given triangle and the inner figure is the rectangle obtained after folding all the three vertices to meet at a point. The three angles fit together to form one straight angle.

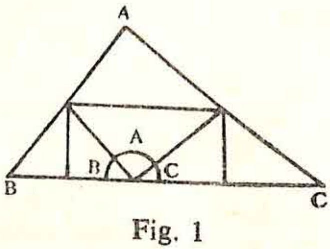


Fig. 1

(iv) The proving of congruency of two triangles, by placing one upon the other practically is also a use of a model.

(v) A triangle can be dissected to form a rectangle, which will enable us to arrive at the formula of its area.

There are innumerable such models to bring different topics home to the students. In fact, there can be a model for almost every proposition and exercise of geometry. Every student should be required to undertake the making of certain models, and the best among them should be selected for the laboratory.

EXCURSIONS

Visits to places of mathematical interest can be of some use. There are a few places which must be visited by every student of mathematics, such as banks, post office, booking offices, insurance, market, auction place, shops, big business concerns, big industrial concerns, etc. These visits can be more useful if they are combined with some preparatory and follow up work.

This is not a laboratory aid, because it takes the learner outside the four walls of the laboratory. This can help laboratory work, if the students form the habit of collecting data and materials on these visits for later use in laboratory work.

COLLECTIONS

Students should be encouraged to collect data, cuttings and materials of mathematical interest. Sometimes there are published very fine graphs in magazines and papers. There are data on percentages, averrages, ratios, investments, incomes, wastages, budgeting etc. The students should be on the look-out for these things, so that such collections can add to the bulk of the laboratory material and are helpful to them and to future students.

FILMSTRIPS

Various filmstrips can be used to give a new colour and attraction to different ideas of mathematics. The resourceful teacher can obtain them from the market. The advantage lies in the fact that they don't burden the mind, can be shown in off hours, and teach the subject in an effective manner. The mathematics teacher should seek the cooperation of the science teacher in using this aid. A short list and titles of some useful film strips alongwith brief remarks on them is given below :

(i) *What is a fraction* ? It explains the basic concept of a fraction as an equal part of an object.

(ii) *Multiplying fractions by fractions*: This filmstrip gives visual examples of multiplying $\frac{1}{2} \times \frac{1}{2}$, $\frac{1}{2} \times \frac{1}{4}$ and $\frac{1}{2} \times \frac{1}{3}$ with many practice exercises.

(iii) *Per cent in everyday life*: It emphasises the social application of "per cent" in the solution of problems involving commission, taxes, interest and discount.

(iv) *Measurement*: It represents the place of measurement in our society and illustrates the use of measures of length, area, volume, liquids, weight, time and temperature.

(v) *Graph uses*: It explains the construction and use of a simple bar graph and shows how to make a graph of two variables.

(vi) *Using the bank*: It explains various services and functions of a bank.

(vii) *What is business*: Understanding the characteristics of business, investments, etc., and an appreciation of the role of trade and business in modern society.

(viii) *Property taxation*: It is on taxation and explains how assessments are made.

(ix) *Sharing economic risks*: It is on insurance.

These filmstrips combine certain distant objects and situations, which cannot otherwise be easily brought and created in the classroom, and present them before the students through a beautiful medium. Before showing them to the class, the teacher should see them himself in order to know their details and other points which have to be emphasised.

RADIO

Broadcasting stations can also help, if the school possesses a radio set and the stations broadcast programmes of mathematical interest. These broadcasts can relate important incidents from the lives of great mathematicians. Some talks by experts can be arranged on the place of mathematics in daily life, and in industry and trade. The history of the development of mathematics can also be a topic, and special emphasis can be laid on the discoveries made by Indian mathematicians. Similarly news pertaining to market rates, temperature and rainfall, broadcast by radio stations can serve as source of collection of data.

SOME SPECIAL INSTRUMENTS

These instruments demonstrate a large variety of simple applications of mathematics to practical field problems. Here are a few examples :

The *angle mirror* is a useful and inexpensive instrument. It is used as a quick, easy and practical means of laying out right angles in the field.

The *plane table* and the *alidade* are used for elementary mapping and surveying.

Proportional dividers can be used to enlarge or reduce maps, diagrams or pictures. It is based on the principle of proportionality in similar triangles.

The combined *hypsometer* and *clinometer* is a simple device for finding angles of elevation and depression and for measuring distances and heights of objects indirectly. It is used for many purposes in simple engineering or surveying and in numerous activities connected with scout work. It is extremely valuable in mathematics classes because of the clear, simple and striking manner in which it illustrates the principles of indirect measurement.

The *level* is another instrument used in finding differences in elevation. It is necessary in contour mapping, general surveying and civil engineering.

The *transit* is an angle measurer and a levelling instrument.

The *slide rule* is one of the most interesting and important of all mathematical instruments. It provides a rapid means of multiplying and dividing numbers, of taking certain powers and roots and of solving proportions. It is being increasingly used as a computing instrument in commercial and industrial work. Its main advantage lies in the convenience and rapidity with which it can be used. Its uses can be learnt easily. One large demonstration slide rule should form a part of the permanent equipment of every mathematics room for students of the high/higher secondary classes.

At least one *calculating machine* for demonstration purposes may be regarded as a desirable equipment for the mathematics department. The students are generally interested in learning when and how it can be used.

By suitable arrangement with other departments in the school, can be made available some such equipment as balances, levers, pulleys, weights, gears, inclined planes, graduated volumetric measures, drawing boards, T-squares etc.

The nature and number of mathematical collections are limited only by the special interests of the students and teacher and by the facilities made available for purchasing and housing the collection.

HOME-MADE EQUIPMENT

Some items can be made by the students and teacher. These pieces of equipment may not be very precise and accurate, but have two notable advantages over those that are produced commercially. First, students always take pride in the equipment made by themselves with the result that their interest in using such equipment is increased. Secondly, they are more likely to understand clearly the fundamental principles upon which mathematical instruments are based. This second advantage is more important than the first.

Among the instruments and items of equipment which can be made by students and teachers may be mentioned the following :

1. Beads, balls, sticks, pebbles, number-picturecards.
2. Almost all the charts.
3. Portraits of great mathematicians.
4. Almost all the possible models.
5. Blackboard instruments such as rulers, protractors, compasses etc.
6. Graphs, budgets, etc.
7. Plane tables, levels, clinometers, proportional dividers etc.
8. Bulletin boards and display cases.
9. Stencils for geometric figures.

But the chief function of the mathematics laboratory is to provide stimulating and worthwhile experiences touching mathematics and its applications. The practice of having students spend time in constructing mathematical equipment is justifiable (a) if it clarifies the underlying principles and increases the students interest, or (b) if it makes available such equipment as is otherwise not available and also if the value of such contribution is at least commensurate with the tune and work required for its construction.

EXERCISES

1. The laboratory of mathematics is no longer a misnomer, it can certainly become an effective medium of instruction. Discuss this statement.

2. Give the list of equipment which you would like to have for the mathematics laboratory meant for secondary classes of your school ?

3. How will you take steps to organise a laboratory of mathematics in your own school ?

4. Justify the need of a mathematics laboratory for effective teaching of mathematics ?

5. Illustrate with examples to show that mathematics laboratory can be helpful in the teaching of a sufficiently large number of topics of mathematics.

6. Write short notes on :

(i) Child's personal equipment.
(ii) Home-made apparatus.
(iii) Use of blackboard in the teaching of mathematics.

CHAPTER NINE

MATHEMATICS LIBRARY AND THE TEXT-BOOK

MATHEMATICS LIBRARY

IMPORTANCE OF LIBRARY

IT IS AN important function of the school to provide the children with extra information when they want it. For this information, they may require to consult reference or general books. Library has come to stay as an established medium of extra information in the schools, especially in those schools which do not aim merely at preparing their pupils for examination. No regular school today can afford to be without a library. It has become an integral part of every good school. One of the important aims of the school is to widen the outlook of the students and to create in them a love for extra reading. The students have not to be kept confined to the length and breadth of text-book knowledge and class-room teaching. Love for extra reading is to be fostered in them. Also co-curricular activities, group projects and individual work on assignments necessitate the existence of an efficient and intelligent library service.

Library service helps in putting progressive methods of teaching into practice. It supplements the knowledge learnt in the class-room. Extra reading makes knowledge complete and comprehensive. It stimulates the students to choose "the best of what has been thought and said". It incidentally educates them for profitable use of leisure. It can help in making a subject lively and interesting which otherwise is likely to remain dry and monotonous if there is complete dependence on the text-book. Moreover, the text-book cannot satisfy the desire of the brilliant students for more and more information. They can satisfy it through supplementary reading only.

The topic of Mathematics Library can usefully be divided into two parts :

(i) General school library.

(ii) Mathematics department library.

GENERAL SCHOOL LIBRARY

In this there should be a separate section for mathematical books. It should contain general and reference books dealing with mathematical subjects.

It is a wrong practice to emphasise the collection of literary books and ignore other subjects including mathematics. Books on mathematics must get a significant place in the general library.

The books on latest techniques of teaching mathematics should also find an important place in this library. It should also contain a selection of good text-books. The text-books must be good, because otherwise there are so many inferior text-books in the market. The selection should be carefully made. It will also be useful to include text-books from other countries. Their advantage lies in the fact that they can be compared with our own text-books from various points of view. The text-books should be of the latest editions.

Though there cannot be many general reference books or recreational books on mathematics, yet as many of them as are available in the market should be provided in the library. Separate books on catch and puzzle problems and short-cut methods may be procured. Some books on the history of mathematics, lives and works of great and inspiring mathematicians may be a good provision.

The books selected must be authoritative and reliable sources of information.

MATHEMATICS DEPARTMENT LIBRARY

It can also be called the subject library. A mathematics teacher may be made the incharge of this library.

This should contain useful reference books of daily use. The books selected should be according to the needs of the pupils and those of the teachers. For the pupils, a collection of current popular books is desirable. Such books should be readily accessible, so that the students may easily consult them during their free time. For teacher's guidance there should be some latest books on methods of teaching mathematics. Only standard books by well reputed authors should be collected. Helpbooks like notes, guides and keys must never be purchased. A number of books on mathematical hobbies and recreational mathematics may also be purchased. The mathematics teacher should try to remain in touch with the various developments that are taking place in the spheres of mathematical literature and the practical teaching of the subject. Whenever there arrives in the market a good book on mathematics it should be purchased and brought to the notice of the students. New books properly selected must be added to the library each year.

A number of copies of a good book should be bought, so that it can be read by everybody. A few copies of the current text-books may be purchased purely for the use of poor students.

Appropriate pictures relating to the history, progress and application of mathematics and portraits of eminent mathematicians will add to the general attraction of the mathematics room and tend to induce a feeling and atmosphere favourable to the study of mathematics.

THE USE OF LIBRARY

Even where suitable books are collected there still remains the necessity of guiding the children in the choice of books or suitable passages in them. The inquisitive child must be directed to suitable sources of information so that after a time he will know how to use the library. The teacher must himself read the book as soon as it is purchased. Only then will he be able to know what is worth reading in a book. He should display on the notice board brief comments about every new arrival. Only a well-read teacher can create love for reading in his pupils.

The best library is that which has empty shelves—empty not because there are no books, but because the students and the teachers have borrowed so extensively.

The number of books possessed by the school library is not so important as the manner in which it is used and the extent to which reading habits have been fostered in the pupils. Inspiration comes from a few masterpieces. It is better to have a few good books than hundreds of useless books.

Some provision should be made in the time-table of every school to have at least two periods in the week for library reading. It should be seen that every student borrows books regularly. There should be some regular assignment and project work which may create a necessity for the students to turn to the library. Some sort of testing out of the material in the library books may also necessitate extra reading. In some schools, it is a very good practice to issue a booklet entitled, "What I Have Read" to every pupil at the beginning of each term. This creates a healthy competition in extra reading.

THE TEXT-BOOK

ITS IMPORTANCE

"Few tools have been so misused as text-books in teaching". It will be no exaggeration to say that text-books have become ends in education. The teachers follow them blindly. They are read out loudly, para by para in the class by each of the pupils in turn; brief explanations and comments are given by the teachers, and all the matter covered is to be memorised by the pupils in the class as well as at home.

The essence of mathematical teaching consists in helping the pupils to discover rules, formulae etc. It is not desirable that pupils learn anything from a book before the matter is taken up in the class. If they do so, the class-room heuristic, analytic or inductive approach cannot be effective. Imagine a situation when the teacher announces the aim of his lesson in the words, "We are going to discover such and such formula", and a student stands up in a corner and offers to relate the formula word by word. So from the point of view of heuristic approach or discovery, text-books are not only unnecessary but far from being desirable.

But the work does not end with the class-room. The teacher is to see that the pupils not only discover, understand and formulate rules, but also remember and readily apply them whenever required. For this, the pupils have to be provided with a record of class-work and also a set of examples for practice. This record can be had by taking notes in the class, but these lengthy dictations and notes are likely to be full of errors. Also taking of notes may prove to be most time consuming. Thus a good text-book saves the time of the teacher as well as of the taught.

It is probably the cheapest and most reliable source of information. It serves as a reference book for the teacher as well.

It is a concise source of material for reviews. It enables the pupils to acquire the needed information speedily. At the revision stage, with the text-book at his disposal, the student can work independently of the teacher.

It helps the students to understand things completely at home from the text-books, even if on certain occasions class-work brings only partial understanding.

If by chance a student fails to attend the school for a few days, he can make up the deficiency with the help of the text-book and catch up with the class.

Home-work and assignment work can easily be given from the text-book.

It specifies the standard to be attained by a particular class. A good deal of illustrative material and many devices are also suggested in the text-book. They are helpful in teaching and correlating the subject with other things. It lays down the order of procedure.

A good text-book furnishes a large collection of well-selected and graded exercises in the teaching of mathematics. After the pupils have learnt a certain new rule or principle, the pages of the text-book may be referred to for application of the rule and homework.

In lower classes, text-books with coloured illustrations provide an incentive to learning and they provide attraction for the young learner. That is why, a text-book for every child has become an essential equipment.

How to Use the Text-Book ?

The text-book is valuable only when it is used properly. The teacher should not feel that his work is confined to transferring the contents of the text-book into the heads of the pupils. The text-book should not be used as the only source of instructional material. It should be used as an aid in teaching, not a substitute for teaching. It is a means and not an end in itself. Its place in teaching can only then be real if the teacher supplements it by his oral exposition, by reference reading and by all his illustrative and objective techniques. It should be followed carefully and intelligently, and not slavishly.

For an average teacher, the text-book is all his stock-in-trade. The greater the capacity, knowledge, professional training and experience of the teacher, the less he needs to depend on his text-book. However, no teacher can afford to work entirely without a text-book. It is not a master to be feared; it is rather a servant to be ordered.

The contribution of the text-book can be increased by creating situations where the pupils have a real purpose for turning to the text-book. Assignment method, Dalton Plan and Project work provide a larger scope for creating such situations.

Essentials of a Good Text-Book

1. It should have the necessary external qualities—i.e., its get-up, paper and printing should be good.
2. In spite of its various qualities, it should be moderately priced.
3. It should be well illustrated. There should be diagrams and sketches at appropriate situations.
4. The language used should be simple and clear.
5. It should be free from mistakes, and the style should be straightforward. In a mathematical text-book even the most ordinary printing mistake can play havoc. The errors have to be very carefully avoided.
6. It should provide sufficient material to motivate the students to solve problems. The problems should be from everyday life.

7. To avoid spoon-feeding, there should not be too many solved examples. The students should get adequate opportunity of learning through initiative and independent efforts.
8. It should be up-to-date as regards the subject-matter.
9. There should be sequence and consistency in the organisation of the subject-matter.
10. The symbols and terms used must be those which are popular so that there may be no confusion. All new terms should be clearly and accurately defined.
11. There should also be a few difficult exercises to challenge the most intelligent students.
12. The author should be a person well-qualified and experienced in the subject. His aims of writing the text-book should be very genuine.
13. There should be sufficient provision for practice and review. It should meet demands of students in ability and interests.
14. The style of writing should be interesting and comprehensive.
15. There should be suggestions to improve study habits.
16. It should offer detailed suggestions for correlation, project work, assignment work and field work.
17. There should not be any irrelevant matter.
18. It should facilitate the use of inductive, analytic, heuristic and laboratory methods.
19. It should conform to the recommendations of important committees.
20. It should satisfy the demands of examinations also.
21. It should be easily available in the market.

EXERCISES

1. How will you organise a mathematics library in your school ?

2. From the point of view of teaching mathematics on heuristic lines, text-books are only unnecessary but far from desirable. Comment on this statement.

3. Mathematics library is another neglected aspect of teaching of mathematics. How will you proceed to remedy the existing conditions ?

4. What are the essential qualities of a mathematics text-book ?

5. Write an essay on the effective use of mathematics library and mathematics text-book.

6. Write short notes on :

   (i) Mathematics department library.
   (ii) Use of mathematics text-book.
   (iii) Need of mathematics text-book.

CHAPTER TEN

DEFECTS IN THE PRESENT-DAY TEACHING OF MATHEMATICS IN SCHOOLS, AND THEIR POSSIBLE REMEDIES

IT SHOULD BE frankly admitted that the present-day teaching of mathematics is far from being satisfactory. This is not due to the drawbacks of any single agency at work. The entire set-up is to blame for this. In spite of the fact that it is taught compulsorily to all the students up to the tenth class, they do not become adept even in elementary arithmetic. A literate person fails to calculate while making payments to a shopkeeper for the articles purchased by him.

Sometime ago a probe was conducted in America to find out the causes of inefficiency of higher technological institutions. Inefficiency was traced to, or attributed to, deteriorating standards of teaching in mathematics. It may be remarked that mathematics plays a vital part in technical professions and atomic researches.

Everybody has a complaint against the teaching of mathematics. It is dull, boring, difficult and useless from the point of view of the learner. "It is too remote from life to interest the student." The teachers complain of excessive work-load and lack of facilities in the form of aids and equipment. Headmasters and managements complain, "It is the fault of the teachers who do not make the students put in adequate labour."

Similarly there are different viewpoints about the necessity of different types of reforms in its teaching. Some take delight only in criticism for the sake of criticism. It is the joint responsibility of all concerned to bring about necessary improvements and changes. The subject has to be popularised at all costs.

DEFECTS AND POSSIBLE REMEDIES

1. TEACHER'S QUALIFICATIONS

It is a common defect in our educational set-up that most of the subject teachers are not adequately qualified in the subjects concerned.

Without proper qualifications and proper training, they fail to do justice to the subject. A teacher may be able to show good examination results in spite of his low qualifications, but even that is not a sufficient criterion to allow him to continue with the teaching of this subject. An adequate, high qualification of the teacher develops self-confidence in the teacher and serves as a source of inspiration to his students. The teacher must be mature in the subject. Professional training should equip him to attain desirable standards in teaching. He must possess real knowledge of, and insight into, the processes of mathematics.

2. TEACHER'S BURDEN

The teacher is overburdened on all sides and, to follow the way of least resistance, he emphasises cram work. He cannot adopt, and prepare for, effective methods, as he has no spare time. His burden does not allow him time to remove individual difficulties. It should be lightened to enable him to show his originality and initiative.

3. TEACHER'S SALARY

His economic position is not good. He remains worried; and a worried teacher cannot give his best to the learners. He runs after other activities to supplement his income. Consequently teaching suffers. He is a frustrated, discontented and half-hearted worker. In these hard days, he must be suitably paid.

4. TEACHER'S ATTITUDE

Maybe, he does not have genuine love for his subject and profession. He may have been forced by circumstances to take to this profession. He remains on the lookout for a better job and leaves the profession as soon as he gets an opportunity to do so. He lacks faith in the utility of the subject, and therefore, cannot create interest among the students. Only really anxious and willing individuals should be allowed to join this profession by introducing a check at the time of selections. A teacher's love for his job and the subject should also be ascertained before giving him his duty.

5. LACK OF PURPOSE

The students do not recognise the purposes behind the study of the topics of mathematics. The destination should be made clear to the students. The particular and general aims of every topic should be emphasised effectively. The teacher has to be careful so that no student ever comes to think that these aims can be attained through easy, soft and amusing work. If the work lacks purpose, it is the teacher's duty to make it purposeful. The purpose should be attractive to stimulate the students to work hard. This misconception should be uprooted from the minds of the parents and pupils that most of the mathematics taught in the school is not purposeful.

6. LACK OF EQUIPMENT

There is a serious lack of mathematical apparatus in the schools. Actual objects have an appeal for the young child. He must be provided with things to see, touch and handle. There should be an adequate provision of concrete material in the class-room. Without it, the subject becomes abstract. The establishment of a mathematical laboratory will remove this defect. Though it is mainly the job of the management and higher authorities, yet the teachers enthusiasm towards its establishment is also wanting.

7. METHOD OF TEACHING

The teacher clings to traditional methods, because these offer the path of least resistance. The powers of thinking, understanding and retention are not thus developed in the students. If the pupils do not show any interest in the subject, it can be created not by blind memorising, but by shifting the methods. The students may grasp anything or not, the teacher does not bother about it. The students forget what is taught to them. There is spoon-feeding and the daily dose of mental work is much more than the student can comfortably swallow and digest. The authorities run after showy results which are obtainable only through cramming. They have no appreciation for good mathematical teaching. There is no emphasis on thought, understanding, initiative, judicious study and power. The remedy necessitates a fundamental change in values. Intelligent understanding should be the teacher's guiding principle. The teacher should not talk too much. He should teach and not preach.

8. RIGOUR IN STUDY

The rules, formulae and set patterns are strictly enforced. The study becomes a tightening belt, not allowing even free breathing. When unfortunately the student cannot adjust himself to this rigidity, he starts thinking that he is unfit. Thus discouraged, he does not make much progress. Class-room atmosphere should be once again charged with freedom and encouragement. The child should be given the opportunity of self-education as far as possible. The teacher should not become a hard task master, but should be a sympathetic helper and guide. Exactness and rigour should not be so important in the very beginning. The emphasis should be on understanding, liking and interest.

9. LARGE CLASSES

It is a general defect. No individual attention can be paid. It becomes difficult for the teacher to establish close contacts with the students. He cannot easily judge the capacities of the individuals. This defect can be removed only by limiting the number of students in each class upto a maximum of forty-five.

10. PRACTICAL ASPECT

The practical and application aspect of knowledge is not generally emphasised. Knowledge given in the class-room is divorced from practical life. The subject loses its appeal, as it is taught in an abstract, dry and uninteresting manner. The affinity between mathematics and life should be discovered and put to use. The students should feel that they are getting something of direct practical value. Mathematics should be taught as a part and parcel of their daily life.

11. MATHEMATICAL LANGUAGE

Mathematical symbols have their own meanings and have their own significance which the teachers generally fail to bring home to the students. The meanings behind these symbols and their historical background should be clarified to the students. A proper use of these symbols should be made in a practical manner. Some assignments necessitating the use may be given.

12. SYLLABUS

Some people say that the syllabus is defective, because it is heavy and lengthy. It might be true in the case of high school syllabus, but the newly introduced higher secondary school syllabi cannot be called heavy. The greater defect of the syllabus is that it does not provide hints and instructions for teacher's guidance. The teacher cannot deal with the syllabus effectively, because most of the details are left to him. The organisers of the syllabus, being experts and authorities on the subject, are expected to suggest the centres of correlation for different topics, the use of aids and devices, the connected practical work, project work, etc. The syllabus has to be thoroughly comprehensive, so that the teachers, all over the area, can get some help from it. The syllabus not only should be a collection of topics, but deal with the actual procedures to be adopted for their effective teaching. Their application and utility in actual life should also be mentioned side by side. The connected plays and activities should be referred to. It may be a bit lengthy, but it must lead to understanding. In that case, the students will not mind a little over-work.

13. TEXT-BOOKS

The traditional style of the syllabus also affects of text-books adversely. The authors have not been able to get rid of dogmatism and traditionalism. The illustrations and problems given in the text-books are divorced from actual life. These have been mainly written on synthetic and deductive lines, whereas the psychology of the child and the nature of the subject require them in analytic and inductive forms. The material is made available in a readymade form which goes against thinking, discovery and originality. The present day books promote cramming and do not lay stress on understanding. They are full of misprints which result in wastage of time and energy. Their style is seldom interesting and impressive. They do not provide suggestions that may facilitate learning. Text-books should give a brief history of the development, possibilities of correlation, applications in practical life, use of aids, plays, activities, projects etc., concerning every topic. They should abundantly present diagrams, sketches, illustrations etc. The arrangement of the subject-matter should both be logical and psychological.

14. THE STUDENTS

There may be some defects in the students of the subject. The subject demands regularity. It is a sequence subject, and if a student is absent even for a few days, the sequence is broken and he fails to comprehend the subsequent steps. An irregular student cannot pull on well in this subject; he becomes weaker and weaker day by day. Similarly, irregularity in home-work also makes the student lag behind.

With the present-day methods of teaching and the criteria of judgment of progress, the students form misleading notions about their intelligence. The crammers excel whereas the intelligent ones may suffer. The present brings frustration for some able students, and there is in store frustration for the crammers in the future. So, in fact, the majority suffers.

Nervous and rash students are also not likely to do well in this subject. The very sight of the examination paper upsets and puzzles them. Moreover, this subject demands whole-hearted concentration which some students may not be easily able to give.

It is the joint responsibility of the home and the school to keep the student regular in attendance and homework. If there is some unavoidable absence, special and separate coaching arrangements for some time are desirable. If a proper approach in its teaching is adopted, only really intelligent pupils will come to the forefront. Only thinkers will survive and crammers will not flourish. To remove their nervousness and confusion, the teacher should try to develop self-confidence in the students. Therefore the habit of learning by self effort should be inculcated.

15. CHILD-CENTRIC APPROACH

Teaching has been subject-centric and not child-centric. The child has been treated as a miniature adult. Knowledge is thrust on him. It has been presumed that all the students of a class have the same capacity; same tastes, aptitudes, power of grasp and speed of work. There has been no consideration of individual differences of the children. The child has been adjusted to the subject, whereas it should have been the reverse. The child should not be subordinated to the subject. He should be given as much as he can assimilate. His interests, likes and dislikes, capacities, and difficulties and aptitudes should be uppermost in the teacher's mind. Let him be given as much as he can assimilate, neither more nor less.

16. LIBRARIES AND LABORATORIES

The organisation of mathematical laboratories is yet awaiting a start. The authorities have not paid any attention to this most urgent aspect to make the teaching effective. The library should offer books of general interest and also books on the methods of teaching. Similarly the laboratory should provide for B.B. instruments, charts, models, instruments and various other materials. The mathematics room should look as such.

17. BAN ON SHORT-CUT METHODS

The use of cut-short methods has been banned totally, but these are of great value and must be employed wherever possible. The quickest, shortest and easiest methods of solving problems should be popularised. The desire to save the time and effort is natural. Quick methods of calculation are much wanted in actual life also. To meet this demand certain special instruments and processes have been invented by mathematicians. The students should be acquainted with these instruments and processes. The teachers should also develop a taste for these short-cuts so that they may introduce them to the students with advantage.

18. EXAMINATIONS

The teaching has been highly influenced by the examinations which are full of defects. The aim is to get through the examination rather than to understand and grasp the subject. When a defective system of examination dominates teaching, the latter is not going to show any improvement. A few selected topics are thoroughly crammed by the students to secure a pass and a major portion of the syllabus is left untouched. There is emphasis on guess papers rather than on real teaching. Then there are heavy failures in this subject. Consequently the students enthusiasm and interest are damped. Examination is a matter of chance more in mathematics than in any other subject. It does not depict the true picture of a candidate's ability and work. A crammer may get the upper hand in the examination, and an intelligent student may not get his due.

This dominance of examination should go. The system needs a thorough overhauling. The annual examination and essay type examination should not be allowed to remain all-important. There should be a true test of intelligence and understanding. Its results should be reliable and valid. At the same time, it should not remain a horror for the young learners. Though a high pass percentage may remain a criterion yet it should have a sound examination system as its basis. Success should no longer be a matter of chance and cramming.

Even after removing the above mentioned defects completely there will be a scope for improvement. Concerted efforts will have to be made for a pretty long time to set right the present deplorable affairs. The best remedy is the devoted teacher. His personality can overshadow all the handicaps. The subject possesses tremendous practical, disciplinary and cultural values. Its importance cannot be overlooked. It should taught as it should be taught.

EXERCISES

1. Is the present-day teaching of mathematics in schools really defective ? If so, what remedies do you propose for it ?

2. Outline the main defects in the teaching of mathematics which usually drive the students away from this subject ?

3. What is lacking in the present-day teaching of mathematics ? What do you propose to do to improve the state of affairs.

CHAPTER ELEVEN

AROUSING AND MAINTAINING INTEREST IN MATHEMATICS—A LIST OF STUDY SUGGESTIONS

PRINCIPLES OF MEANINGFUL LEARNING
MATHEMATICS CLUB

TO AROUSE AND maintain the student's interest in mathematics, is a major problem for the teacher. He knows that loss of interest is one of the principal causes of student failure. Students work most effectively at task in which they are genuinely interested. Students, as a rule, readily become interested in things which are new or exciting, for which they can perceive practical values and which involve puzzle elements or elements of mystery. Their interest is easily caught by anything new, but such interest is fleeting. It is easier to interest students in their work than it is to keep them interested after the novelty of the work has worn off.

The elements of novelty, usefulness, and sheer intellectual curiosity are the primary stimuli for the awakening of interest. The work should present a continual challenge, but it should be a challenge in the real sense and not merely drudgery at meaningless, difficult tasks. Interest in the subject can be effectively aroused and maintained by numerous special devices and activities.

INTELLECTUAL ACTIVITY

Let there be no presumption that the students are intellectually lazy. The student's interest can definitely be stimulated through a challenge to his curiosity. 'Mathematical situations lack, of course, the lurid "human interest" of the ordinary mystery novel, but they do not lack the essential curiosity provoking possibilities.' Mathematics exhibits fully the power of man to think consistently and logically. In the case of some students, this value of mathematics creates an appealing interest of highest importance.

A natural motive for man's fight against his environment is his desire to understand and consequently to control it in some measure to suit his own purposes.

This intellectual activity is governed by three motives : (1) thirst for knowledge, (2) love of truth and beauty, (3) desire to interpret and control environment. Of course, the cultivation of these may require certain discriminating abilities and maturity of thought which usually result from the application of more elementary means of motivation. As the student progresses in learning, these motives can become increasingly effective in controlling his activity. To make them controlling factors in the learning process should be one of the principal aims in teaching.

APPLICATION TO OTHER FIELDS OF STUDY

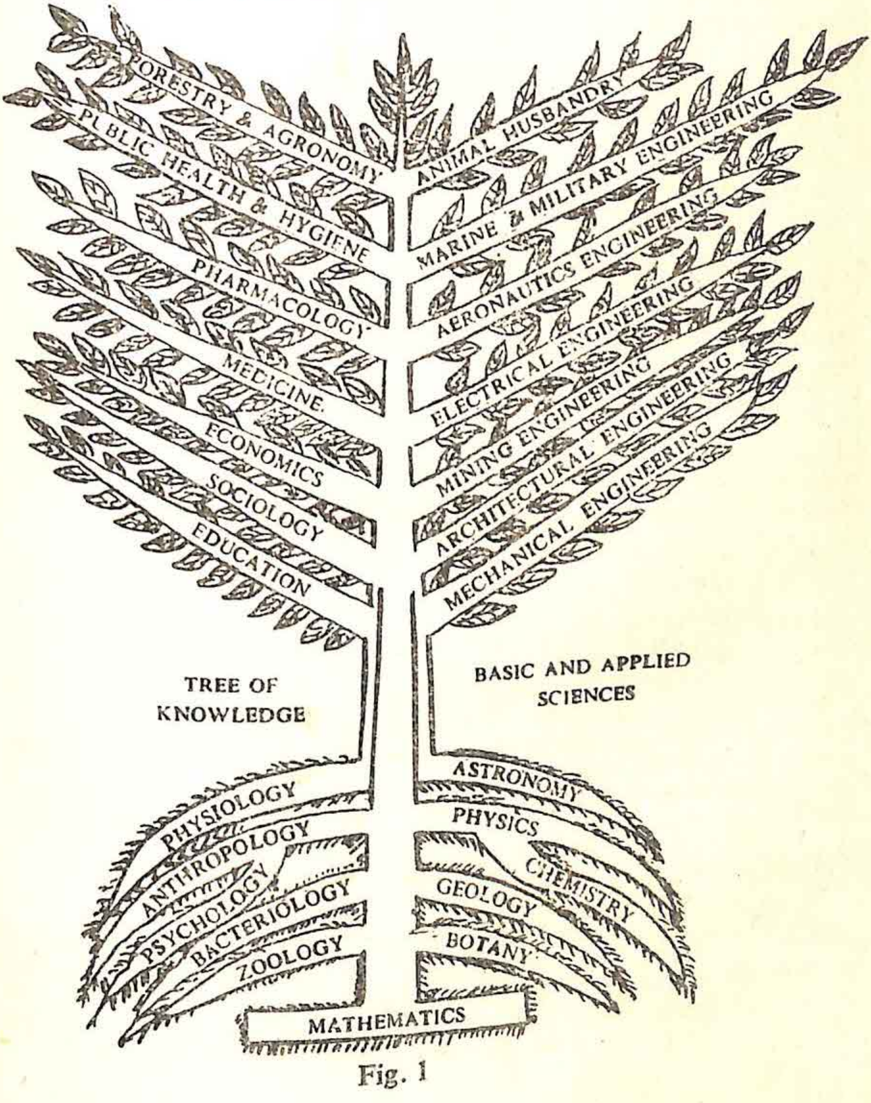


Fig. 1

Very often a student's interest may be aroused by showing the applications of mathematics to some other field in which the student is already participating actively. The basic importance of mathematics in relation to other fields of study has been nowhere more emphatically or more strikingly portrayed than in the mural entitled 'The Tree of Knowledge' by John Norton (Fig. 1).

The foundation of the 'Tree of Knowledge' is composed of the basic and applied sciences and represents mathematics as the principal life-giving root. Other roots, stems and branches representing the various basic and applied sciences spring from it. The dependence of different sciences upon mathematics has been explained so comprehensively in the chapter entitled "Relationship of Mathematics with other Sciences" that it is hardly necessary to dwell upon it again. No informed person would question its instrumental value in this connection.

APPLICATION OF MATHEMATICS TO PROFESSIONAL FIELDS

Mathematics is now coming to be recognised as a necessary part of professional equipment in such fields as anatomy, physiology, psychology, medicine, geology, industry, engineering, commerce and agriculture. 'Necessity is the mother of invention.' Further, the urge to earn a direct livelihood often drives individual to seek "pasteurs new", to venture where he would never care to tread in normal circumstances. An efficient study of mathematics raises the student's competency in different vocations. By pointing out its applications, teachers can render valuable service in guiding students and at the same time in stimulating their interest in mathematics itself.

PRACTICAL VALUES

In addition to these vocational values, numerous other practical values are extremely useful for the average citizen. A student should question both the theoretical and the practical value of any particular place of mathematics. The 'need' or 'utility' consideration should guide the student activity in learning mathematics. Though it is not always possible at a given stage to fully appreciate an exposition of such values, a partial explanation may suffice from the viewpoint of the student.

CULTURAL VALUES

There are many important aspects of the world which only mathematics can interpert to the citizen. A reference to the historical background of different mathematical terms and ideas gives a good start for learning them. The teacher should spare no pains to make the subject-matter attractive and pleasant. Students feel encouraged to work harder if accomplishment is accompanied by pleasure. The cultural values received are generally proportional to the pleasures derived. The students should be kept aware of the nature and the universal applicability of mathematical methods. This will provide strong motivation for study.

RECREATIONAL VALUES

The aim of teaching mathematics for fun and pastime also should receive some attention. The employment of mathematical puzzles, contests and games lends much interests and zest to the courses. An active mathematics club is one of the sure signs of student interest. Maximum interesting project work should be introduced. The programmes of mathematics clubs may cover a wide range of topics including history of mathematics, biographical sketches, interesting anecdotes and applications of mathematics to other subjects and fields of activity. Organisation of a mathematical workshop can provide a combination of recreation, activity and learning.

PRACTICAL WORK

Handling of material and learning through experimentation should be permitted and arranged as frequently as possible. Practical presentation of matter should always precede its abstract form. Preparation of aids and applications based on mathematical topics should be a regular activity of the learners.

THE PRINCIPLE OF CHANGE

Monotony should not be allowed to settle over the class-room atmosphere. A topic should not be continued for too long a period. No aid or device should be employed over and over again.

PHYSICAL CONDITIONS FOR STUDY

Room temperature of about 68°F is considered best for study. All necessary study accessories such as paper, pencil, ruler, compasses, notebooks, mathematical tables etc., should be at hand. Physical discomforts should be eliminated. Distractions should be removed or avoided. Work should be started promptly at a fixed time and continued without interruption until it is completed or until a short exercise period or a rest period is needed. At home, the student should be given a quiet room, well lighted and ventilated, with a chair and table for convenient work.

PSYCHOLOGICAL CONDITIONS FOR STUDY

1. Avoid unnecessary distractions and interruptions.
2. Cultivate concentration.
3. Keep the mind alert.
4. Develop self-confidence.
5. Practice critical thinking.
6. Proceed systematically and thoroughly.
7. Analyse difficulties.
8. Learn accurately the fundamentals.
9. Clarify thought by writing.
10. Verify results.
11. Be prepared for discussion.

The above mentioned conditions, if provided and enforced will surely lead to better results.

A LIST OF STUDY SUGGESTIONS

Students often follow wasteful and inefficient procedures in studying mathematics. Sometimes they fail to get at the first step of a process. They do not develop self-reliance. They proceed unsystematically and cannot concentrate. They are careless in their reading, in their listening and in their written work. The teacher must enable the learners to appreciate and grasp the specific procedures which are helpful in the study of particular parts or aspects of the work.

This list of suggestions is expected to be of help to students and teachers for this purpose :

1. Try to derive pleasure through a sense of achievement by solving mathematical problems successfully.
2. Fix time and place for the study of mathematics and be regular in your work.
3. Get down to work at once. Do not daily.
4. It challenges your intellect; meet is squarely.
5. Pay to it deep and sustained attention.
6. Do not allow avoidable interruptions after you have started work.
7. Work independently as far as possible.
8. Work at your maximum speed.
9. Plan thoroughly before you start.

10. Understand the assignment clearly.
11. Read the problem carefully and grasp its meaning. Be sure that you understand clearly what is given and what you are expected to find or prove. Keep these things in mind throughout your work.
12. Take sufficient time to think.
13. Even if you have to copy something, be sure that you do it correctly.
14. Try to enlist questions that are likely to arise and be ticklish. Often the solution will flash across your mind.
15. Do not lose hope; at least try to find out what your real difficulty is.
16. Work carefully, for mistakes can be more easily avoided than detected and corrected.
17. Read thoughtfully. Superficial or careless reading does not pay in mathematics.
18. A problem is generally made up of a series of steps. Do one step at a time.
19. Be mentally alert, aggressive and confident.
20. Be critical of all statements or conclusions whether they are your own or someone else's. Be very particular about statements that are not adequately supported by reasons.
21. If you do not know how to begin, consult your text-book or your teacher.
22. Prepare a list of all new words and concepts and try to learn them at once.
23. Do memorise important rules, formulae and facts, but only after understanding their meanings.
24. It is easier to memorise statements and formulae as whole than to memorise them by parts.
25. In memorising formulae, loud reading is helpful.
26. Write neatly. Be very particular about figures. Write in neat rows and columns.
27. Form the habit of expressing verbal statements in symbols.
28. Form the habit of verification. Always check up your work.
29. In numerical problems, form the habits of making a preliminary estimate to serve as a rough check up on your answer.

30. Correlate and compare exercises in algebra with similar exercies in geometry and arithmetic.
31. Lay special stress on the application of rules and formulae, in order to fix them permanently in your mind.
32. Draw diagrams and sketches wherever possible. They often help in understanding and solving problems.
33. Do not have books open or a pencil in hand while listening to a discussion in class. Pay it undivided attention.

PRINCIPLES OF MEANINGFUL LEARNING IN MATHEMATICS

MEANINGFUL LEARNING

Before we try to understand the concept of meaningful learning, it will be better here to know once again what learning stands for. The definition given by R.S. Woodworth appears befitting, which runs as "An activity may be called learning in so far it develops the individual in any way good or bad and makes his environment and experiences different from what it would otherwise have been."

Learning can produce both good and bad developments in the learner. But the learner, his guardians, his teachers and the society in general want the process of learning to lead to good results and healthy outcomes. That learning is to be avoided, curbed or replaced which is likely to prove harmful in any way. Not only informal learning, but even formal learning can be injurious to the learner and his well wishers in some way. The schools and the teachers have to be cautious in this respect. Child comes to school for better experiences, where the situation should not be worse for him. Every activity of the process of learning should be governed by certain aims and objectives. There cannot be any thing of wasteful, useless, aimless or meaningless nature.

Meaningful learning is therefore that learning which is oriented towards good experiences and outcomes. In it there is no place for meaningless and harmful experiences. It must ensure positive results. It is constructive, productive, purposeful and progressive in nature.

Meaningful learning in mathematics can consist of the mathematical experiences of the following type :

1. Which are helpful in mental, emotional and social development.
2. Which have utilitarian, practical and behavioural values.
3. Which are useful in learning higher and advanced aspects of the subject.

4. Which are helpful in the proper learning of other subjects and activities of the curriculum.
5. Which stimulate and maintain interest in the subject.
6. Which lead to the development of proper attitude towards the subject.

PRINCIPLES OF MEANINGFUL LEARNING

There are many principles of meaningful learning in general and in mathematics in particular which are being discussed here :

1. INTEREST AND ATTENTION

In order to make learning in the subject meaningful, it is necessary that the students are motivated before they are made to learn anything. A strong interest should be aroused and whole-hearted attention to the extent of concentration should be secured.

2. DEFINITE AIM

The aim and objective of every topic, sub-topic or lesson should be made definite and clear before them.

3. PRINCIPLE OF UTILITY

The aim of every topic or sub-topic can be made more definite if its utility in child's present or future life is thoroughly explained. If they come to realise in the very beginning that the material to be learnt will find wide applications, then their learning will be need-based and hence meaningful.

4. IMPORTANCE OF LEARNING ACTIVITIES

The learnig activity should look important, significant, relevant, life-oriented, and stimulating to the learners. This will further ensure meaningful learning among them.

5. MOTIVATION

The interest as principle has been stressed earlier. Motivation is the basic activity for creating interest. Various means such as aids, relationships, historical references, anecdotes, possible applications, games can be adopted to provide motivation.

6. PROPER ATTITUDE

There are attitude like heuristic and problem-solving which are basic to meaningful learning. When such attitudes are developed from the very beginning the students enjoy learning mathematics. To them mathematics would never appear dull or dreary.

7. PROPER METHODS

All other principles of meaningful learning will be defeated if the teacher fails to adopt befitting methods of teaching and the students on their part fail to adopt suitable methods of learning. It is obvious that methods like heuristic, analytic, induction and problems are recommended to the teacher and those like self-effort, understanding, intelligent practice, and problem-solving are recommended approaches for the learner.

8. CORRELATION

When correlation of mathematical knowledge with life, other sciences, and activities is stressed, the learner can appreciate the meaningfulness of his learning.

9. PROPER USE OF MENTAL FACULTIES

It pays to exercise the thinking and reasoning powers of the learner as against the cramming, memorisation and retention abilities. The learning becomes meaningful if it is saved from mechanical process and blind cramming.

10. LAW OF EXERCISE

In the learning of mathematics there is need for regular and persistent practice. The student should never lose patience, hope and self-confidence. Thus his learning will become meaningful and fruitful for him.

MATHEMATICS CLUB

Like other subject clubs and societies, there is a genuine place for a mathematics club in a school of today. Broadly speaking, it will be organised in the same way or on the same pattern as other subject clubs or even co-curricular societies. The mathematics club is no longer a vague idea, it has become a reality in many progressive schools. This club can play a significant role in the total programme of a school. It can serve a number of purposes given on page 171.

ITS PURPOSES

1. It can provide a forum to those interested in mathematics for mathematical activities.
2. The students get opportunities of mathematical hobbies, recreational mathematics, mathematical projects, mathematical games, mathematical discussions and debates, and mathematical innovations.
3. The club will be a medium of developing students' interest in the subject.
4. Its activities will relate knowledge with activity and life.
5. It will provide to the students an opportunity of listening to experts and teachers from outside. They can be invited through this agency to address the students on different topics.
6. It initiates and develops mathematical expression among the students.
7. It can be an agency for providing intra-school and inter-school mathematical competitions.
8. It can organise excursions and visits of mathematical value.
9. It can provide an opportunity of extra reading of mathematics and reading of general mathematical literature.
10. It can be an agency to prepare and display mathematical illustrations.
11. It can help in the decoration and equipment of the mathematics room or laboratory.
12. It will be a medium of exchange of mathematical information, experiences, experiments, and innovations.
13. It will provide to the students opportunities of organising various activities and thus develop their organising ability.
14. It will provide to some the opportunity of leadership and to many the opportunities of cooperation, joint responsibility and active participation.
15. It can arrange mathematical film-strip shows and film shows for its members.
16. It can look after the mathematics section of the school magazine.
17. It can cooperate with other societies in the school for various functions.

ORGANISATION OF THE CLUB

The head of the institution will be the patron and the senior teacher of the subject will be its incharge. All other teachers of the subject will be staff advisers. Other subject teachers interested in mathematics will be associate staff advisers. The students of the subject will be the members. If the number is very large, there can be two clubs—one for the senior students and the other for junior students. The members may elect the necessary office bearers from amongst themselves to discharge various specific duties.

ACTIVITIES OF THE CLUB

The list of functions in a way incorporates the activities of the club. Still for greater emphasis the activities may be enlisted here separately :

1. Mathematical games, hobbies, projects, experiments, data collection, discussions and debates, and innovations.
2. Lessons and lectures by prominent teachers from outside.
3. Lectures by experts of allied subjects and professions.
4. Organising mathematical competitions.
5. Organising excursions and visits of mathematical value.
6. Preparing mathematical aids and illustrations.
7. Decorating and equipping mathematics room and laboratory.
8. Organising general and extra reading in mathematics.
9. Arranging mathematical shows, exhibitions, etc.
10. Mathematical articles for the school magazine.
11. Relating the teaching of mathematics with other subjects, life activities and environment.
12. Looking after the interests of the subject.
13. Collaboration with other clubs and societies in the school functions.

EXERCISES

1. How will you create interest in the teaching of mathematics in the existing conditions in schools ?
2. What devices can possibly be used to arouse the students interest in mathematics ?

3. Mathematics is considered a dull and dreary subject. What steps would you take to counteract this impression ?

4. The teacher of mathematics has special responsibilities towards the use of certain ways and means to make the teaching of his subject effective. Comment and elucidate.

5. What do you understand by meaningful learning in mathematics and what are the various principles of meaningful learning ?

6. Enlist and discuss the purposes and activities of mathematics club in schools.

7. How will you employ the medium of mathematics club for the benefit of students ?

8. Write short notes on the following :

   (i) Need for a mathematics club in schools.
   (ii) Study suggestions for the students of mathematics.
   (iii) Principles of meaningful learning in mathematics.
   (iv) Organisation of mathematics club in a school.

CHAPTER TWELVE

GIFTED STUDENTS AND BACKWARD STUDENTS

GIFTED STUDENTS

THESE ARE THE students with exceptionally high I.Q. They are a very small percentage in any normal population. There may be none such student in a school or there may be at the most one or two such students, even in a big school. How to do justice to a gifted student who is the only student of his calibre in a big section with fifty or more students ? This is a big question before the teachers and educational administrator.

The gifted students can be rightly called the cream of the population. They are the potential leaders for different walks of life. If given proper education and provided suitable opportunities, they can make significant contribution to the overall progress of the society. In a way, they are the real asset of any society. Their worth cannot be determined by any known means of measurement. Most of the future politicians, administrators, scholars, discoverers, inventors, poets, builders, reformers, philosophers, planners and teachers have to be drawn from them. The gifted is not a gift for himself alone, he is a gift for the society as a whole. If given a chance, he is going to do something extraordinary and remarkable. If neglected, he will be a loss to himself no doubt but it will mount to a far greater loss to the society. Ordinary people generally spend their lives in ordinary ways, but the gifted ones devote themselves to higher tasks of breaking new grounds, and leaving behind their footsteps for others. They are the stuff which is most precious and which deserves special attention in the matter of education.

LOCATING THE GIFTED

Before we plan any programme of special education for them, it is necessary to search for them and to locate them in the vast sea of student population. They are expected to be scattered all over in very small numbers. Preliminary search can be based on achievement testing which is normal feature of the schools. Further confirmation of the cases can be made by intelligence tests. There are some other prevalent schemes known as talent search programmes organised by NCERT and other agencies which can also help in locating these cases.

ENRICHMENT PROGRAMME FOR THE GIFTED

After they have been located, the gifted students should preferably be collected at one central place in a district or a region. If they are left in ordinary schools among the companions mostly of average or below average calibre they will not be able to do their best. Among ordinary companions they will be at the top without much effort and will not face any competition. Also in the class most of the students will be of average ability and the teacher will teach with that level of students in mind. Teacher's teaching addressed to average students will not be of much interest or benefit to the gifted student. It will be better if they are collected at one place in an ideal school, brought into company and competition with classmates of their level and are placed under the care and charge of really competent teachers. Such like schools can also be made residential where all-round development of these students may be ensured. The society or state can bear the expenses in the case of poor students of this type. The school meant for them can also be equipped adequately to cater to their varied educational needs. A truly academic atmosphere can also be created. Apart from all this the following arrangements may be made to do full justice to them in the matter of educational opportunties.

1. An enriched syllabus may be proposed for them to provide to them extensive and intensive education.

2. Really challenging, ingenuity demanding, initiative demanding and creativity-oriented tasks may be proposed for them.

3. Their learning should not centre around the text-books but should be expanded beyond with the help of supplementary readers, reference books and general literature.

4. Their system of evaluation should also be more reliable and sophisticated in order to remove from it chance factors and errors of various types.

5. They should be provided with rich, efficient and prompt library service.

6. They should be taught by the methods like heuristic, induction, analysis, project and discussion.

7. Their self study should be exercised to the maximum by proposing to them intelligent assignments.

8. They should be largely left to themselves to solve mathematical problems and should be made to enjoy full self-confidence.

9. Mathematical problems proposed for them should not be routine type or mechanical in nature. They do not need repeated problems of the same rype. Every problem meant for them should be a type in itself.

10. They should be made to realise that mathematics offers a real challenge to their genius.

11. They should be given opportunities of collection, tabulation and interpretation of mathematical data.

12. In every possible way, they should be provided with quality education.

BACKWARD STUDENTS

There may be two types of backwardness :

(i) General backwardness, and
(ii) Particular backwardness.

GENERAL BACKWARDNESS

This means all-round backwardness. It is almost inherited and is due to a low I.Q. which may be between 50-70. We can at the most raise it by 7 or 8 points. The removal of this backwardness is a joint responsibility of all the teachers in the school. Of course, the mathematics teacher will be required to pay special attention in the case of his subject. If the case of the child is a hopeless one, it should be guided sympathetically to take to some manual type of vocation.

PARTICULAR BACKWARDNESS

This is the case, which is backward in mathematics but fairly intelligent in other subjects. The child is otherwise intelligent, but is weak in the mathematics. Removal of particular backwardness is easier than removal of general backwardness, because the former is hopeful case. His being intelligent in other subjects, is a hopeful sign. His achievement in other subjects is a source of confidence and inspiration for him. This backwardness is the sole concern of the mathematics teacher. Acute cases may have to be treated in separate classes, whereas border cases may be tackled in the class-room itself.

Before adopting any means to remove the backwardness, the teacher must be clear about its causes which may be many.

CAUSES AND REMEDIES

1. There may be some physical cause, such as poor eyesight, defect in the hearing organ, stomach trouble, headache or any other physical ailment, which does not allow the child to concentrate on his studies. Mathematics needs special concentration. The remedy of almost all these causes lies with the doctor or physician, but some sort of physical exercise may also help the child. The teacher has to refer the child to the school doctor, some other medical expert or the school PTI according to the nature of the case. He has to try to make available all the possible remedies. If it is a case of serious physical deficiency, the teacher should try to persuade the parents to get their ward admitted to a special institution for the physically handicapped children of that type.

2. The backwardness may be due to some mental causes. These causes may be inborn and environmental. The child may have low I.Q., some mental ailment, mental dissatisfaction, domestic problem, mental conflict, sense of insecurity, inferiority complex, lack of interest in mathematics or a dominant interest in something else. A case of simple mental problem can be tackled with some chances of success by the teacher. But complicated cases will have to be passed on to a psychologist or a psychoanalysts for thorough diagnosis and treatment. An attitude of love and sympathy and preparedness to help the children on the part of the teacher will always be helpful in the treatment of such cases.

3. A distaste for the subject may be another cause. This distaste may either be natural or acquired. If it is inborn, the teacher's efforts may go waste. But if it is an acquired one, it may be mostly the teacher's fault. If he fails to develop a feeling of attachment between the child and himself, the only outcome is the child's distaste for the subject. Heaviness of the syllabus, toughness of the subject and difficulty of its problems also tend to produce this distaste. A genuine taste for the subject will be developed through the teacher's patience and persistence. He should never be in a hurry to pronounce a case as hopeless and backward. If by chance a child fails to satisfy him once, the teacher should not get disappointed. He should never allow the idea 'once backward always backward' to enter his mind.

4. Sometimes the child develops some doubts about the fundamentals of the subject, and these doubts hinder his progress throughout. The teacher should be very careful while teaching these fundamentals. He should not hesitate to explain them over and over again if required. Only a clear understanding of these fundamentals can build a sound foundation for mathematical learning.

5. Influence of the home may also create distaste and backwardness. Some parents unintentionally provide negative suggestions to their children. Some of them are in the habit of saying that they

never liked mathematics or they never wanted to study it, or were never able to pass in it or that failure in mathematics has been a tradition of their family. Even the educated commit such mistakes. These pronouncements are likely to have an adverse effect on the child's attainment in the subject. Parents must be made conscious of this adverse effect and their duty in this matter.

6. The teacher's unbalanced behaviour may also be one of the causes. If he is very lenient, some of the clever and mischievous students may get undue advantage of it and may become backward due to continuous inattention and non-seriousness. If he is very strict and gives heavy punishment unsparingly, some of the feeble-minded students may get disheartened and discouraged. They may start disliking the teacher and consequently the subject. The teacher should never forget that his behaviour is very important.

7. The change of school or even the change of teacher may not suit some of the students. Whenever the change of school is unavoidable, the parents and the teachers have to remain on guard till the child's complete adjustment with the change. Changing the teacher in the middle of a session should be avoided as far as possible.

8. The subject teacher's bad reputation in the school and neighbourhood may also affect some of the students adversely. The teacher must earn respect and reputation. He should be a source of inspiration to his students in every respect. He should carefully maintain his scholarship and character.

9. A part of the students' apathy towards the teacher's most favourite method of teaching may result in their backwardness. The teacher should not always stick to the same method. Students like 'newness and novelty'. The teacher should always be prepared to adjust his method to the learner's likes. He may have to devote extra time and attention to the student who is lagging behind. Slow learning deserves the teacher's individual attention and help. The teacher should so conduct himself that the slow learners never feel that they are being disowned or being left behind.

10. Of these students, some may need more practice and drill than others. The teacher should not overlook their need. The intelligent one may pick up an idea after solving only one problem, whereas the slow learner may need solving two or three problems to grasp it. The teacher has to adopt a via-media.

11. When the child does not get sufficient time at his home for homework, he is likely to lag behind. Regularity in homework should be ensured. The parents should give full cooperation to the teacher in this regard.

12. If the child remains unavoidably irregular or absent for a long time, there are chances of his falling a prey to backwardness.

The facility of extra coaching for some time should be provided to such students so that they can fill up their gaps and catch up with other students of the class. Such temporarily backward students should not be left to themselves. If need be, extra classes may be arranged for them. The provision of extra classes made in most of the schools near the examination, enables many backward students to get through.

13. Defective hand at writing and geometrical constructions may be the cause of unsatisfactory performance in certain cases. Their weakness does not therefore pertain to mathematics. The cooperation of language teachers may be sought in their case. The teacher should make it a point to insist on neatness of hand and construction in such cases.

EXERCISES

1. How will you discover gifted students in mathematics, and what will you do to help such cases ?
2. What are the possible remedies to remove a student's backwardness in mathematics ?
3. What are the main causes of backwardness in mathematics ? What are the corresponding effective remedies ?
4. Backwardness in mathematics is not always due to the low I.Q. of a student. Comment and discuss.
5. How will you discover particular backwardness in mathematics, and what will you do to pull up such cases ?
6. Describe and discuss enrichment programme in mathematics for gifted students.
7. Write short notes on the following :
    - (i) Gifted student is at a loss in ordinary classes.
    - (ii) Need for special attention to gifted students.

CHAPTER THIRTEEN

TECHNIQUES OF TEACHING

THERE ARE A number of techniques which can be effectively used for the teaching of mathematics. Some of them are : oral work, drill work, home work, assignments, self-study, group work, review and supervised study. They are being discussed here.

ORAL WORK

What is oral work ? It is the work which is done orally without the help of written work and record. It can be defined as mental work, wherein a problem is solved orally or mentally, without the use of pen and paper. As a matter of fact, much of mathematical work has to be done mentally and many tables have to be learnt by heart. It precedes written work. Of course, when calculations become more complicated and heavy, writing is used as an aid. As mind cannot carry out those calculations orally, written work becomes indispensable. But in all respects oral work is the backbone of not only written work, but also overall performance in the subject.

FUNCTIONS OF ORAL WORK

1. It has an appeal for the eye and ear, and this appeal is liked by students.

2. With the help of oral work, some time can possibly be saved.

3. It is very suitable for securing attention of students and for making them concentrate. A few oral questions can make them alert and active.

4. It is useful in everyday life, because in actual life pen and paper are often not available and one has to do certain calculations mentally.

5. It helps in elucidation and illustration. The same idea can be effectively illustrated through a sufficient number of oral examples or

questions presented in quick succession without much loss of time.

6. It removes shyness of pupils through oral expression.
7. It quickens wit and sharpens intelligence.
8. It is a good mental exercise, because it develops alertness, readiness of mind, quick hearing and quick thinking.
9. It is an effective means of maintaining discipline.
10. Briskness in oral work is stimulating for the students.
11. In case of an error, it helps in its mental cancellation.
12. It is interesting and effective especially in the initial stages of every topic.
13. It is an easy way of testing previous knowledge.
14. It helps in connecting the introduction of a new topic with the previous knowledge of the students.
15. Oral questions enable the teacher to judge whether the students are following the lesson or not.
16. Recapitulation and revision can also be done by well-framed oral questions.
17. The teacher can throughout remain in touch with the class with the help of oral questions.
18. It provides variety in the otherwise monotonous routine of the class-room.
19. If properly used, it can provide motivation.
20. It develops accuracy, exactness and precision.
21. It encourages healthy competition.
22. It builds strong foundation for later written work.

Its Place

It should be crystal clear from the above mentioned functions or values of oral work, that it has an appeal for the eye and the ear of the child. This appeal has greater value than insistence on written work in the very beginning. Oral work gives a good start to every topic. All new processes and methods should be introduced, and fixed in the child's mind, orally. Any individual difficulties can be effectively removed by oral work. In certain cases, where repetition is necessary, oral work takes the form of oral drill. Oral recitation or oral repetition of tables is often a source of enjoyment to students. Oral mastery of every new idea by students must be the teacher's first concern. Every new concept must be given a sufficient oral drill. Oral work is the mainstay of every lesson.

WRITTEN WORK, ITS COUNTERPART

Oral work can only give a start, and new concepts have to be ensured and ascertained by putting them in black and white. Oral work must be supplemented by written work. Oral discussions and mental images are not always enough and cannot possibly be retained for long. Mental work has therefore to be combined with written work. Oral work has to supplement written work and not to supplant it, because efficiency in written work is the ultimate aim. Oral work must contribute to success in written work. Over-emphasis on oral work will also be a harmful attitude. As a habit forming measure, both oral and written work should be included in every lesson. In certain cases, some students may not gain substantially from oral work alone; their interests will be served if oral work is followed by written work. Written work is, in a way mental confirmation of information acquired orally. It is an extension of oral work. It affects accuracy and precision more effectively. It has got its own place and utility. Whereas oral work forms the beginning, written work forms the end in the learning of a new process. The saying, "Reading makes a full man, conference a ready man, and writing an exact man," is to some extent applicable to the teaching of mathematics also. The teacher may adopt a flexible division of time to provide for different phases of work, namely :

1. Oral fundamentals.
2. Written fundamentals.
3. Oral problems.
4. Written problems.

In no case should oral work be divorced from written work and vice-versa. They are companions. They work best in combination and partnership.

DRILL WORK

DRILL AND ITS NEED

There are four chief devices of fixing an impression on the minds of pupils– drill, review, asking questions, and describing outstanding problems still to be solved.

Drill is one of the most widely used devices and practices in our schools. It is a serious activity—it provides opportunity of self-improvement. One cannot expect to achieve speed and accuracy in solving mathematical problems without it. The basic facts and operations of mathematics have to be memorised through sufficient drill, but there should be no mechanical cramming. Practice should be done with understanding and with a clear idea of the objectives and goals.

HOW TO MOTIVATE DRILL

A spirit of play should be introduced in order to make the process of memorisation less tiresome. This can be achieved by making use of number games, flash cards, self-direction material and application of ideas in various situations at home and school. It should often be indirect and disguised.

A FEW PRINCIPLES UNDERLYING EFFECTIVE DRILL

1. Moderate initial practice and systematic review are superior to overlearning.
2. The material to be memorised or drill upon should be meaningful.
3. Drill exercises should be short and distributed over a period of time.
4. Much of the drill work should be individualised.
5. Make success in drill possible and let the students feel it.
6. Drill should be varied; it should be given in various forms, otherwise it will become monotonous.
7. Drill should be properly and sufficiently motivated; and once initiated, it should be well-organised.
8. Drill assignments should not be made tasks; the students should be enabled to take pleasure in them.
9. The achievement of the learner in drill should be frequently tested.
10. It would be beneficial to introduce the elements of play, competition and group work in drill.

Effective drill, not only develops knowledge and skills, but also is a means of maintaining good habits when they have been once established. It is always intelligent and purposeful; it is economical of time, whereas unintelligent drill is wasteful and purposeless.

HOME WORK

ITS NEED

School time is insufficient to exhaust everything provided in the curriculum of mathematics. Homework has to be given regularly to provide for application and practice and to supplement class-room teaching. It becomes all the more important and necessary in view of the heavy load of work as compared to the time available.

Some educationists are of the view that home work should not be given to the students. They feel that if the school time is used sincerely and efficiently, homework does not remain necessary. Under normal conditions of work, the child does a good amount of work during school hours. Giving homework means creating in the child school mania at home also. Others are of the view that five hours' work at school is insufficient and so it must be supplemented by some work at home. It is not the homework that is objectionable, but its nature and amount which invite criticism. Some of the teachers have a mania for homework. They over-burden the students with it. It may even affect adversely the health of the students. In quite a few homes, the conditions may not be congenial for the child to do the homework. It is educationally unsound to load the young child, with homework, especially when he loves to play and enjoy the time rather than to do serious work of calculations and solving sums. Heavy homework snatches the leisure-time pleasure of the young child. A large amount of homework may create a distaste for school in the child. The constant anxiety of doing homework may adversely affect the progress of the studies. It may lead to a disgustful reaction towards the subject which may be transferred onwards to the teacher.

But the importance and need of homework cannot be minimised, if it is of a right nature. It utilises the leisure time of the children, which otherwise would have been wasted. It may establish the habit of working hard and that too regularly. It supplements and confirms class-room work and is a practical means to cover comfortably the lengthy and heavy syllabi. It may serve as a link for parent-teacher cooperation. It provides to the child opportunity of independent work.

How to make it More Useful

1. It should be well graded. Its nature and amount must be determined by the individual's capacity and interest and not by the whims of the teacher. It should be individualised as possible.
2. It should not be considered as a sort of punishment.
3. It should not be given by the teacher in a disturbed state of mind.
4. Home conditions of the individual children must be borne in mind, while determining quantum of homework.

5. Homework time-table should be prepared in advance. Each subject must get its due share of homework. Total amount of work allotted on a particular day should not exceed reasonable limits.
6. Homework should be duly checked and corrected. If it is not checked, the students may fall into the bad habit of evading it or copying it.
7. It may not necessarily be of academic nature. It may be of practical type. Hobbies connected with the subject, such as making a model, collecting some data or solving a riddle, also come in this category.
8. It should be assessed as a part of the over all assessment of a student.

In the end, it is suggested that homework should appear enjoyable, useful and attractive. It should suit the individual speed and interests. It should be intelligently planned and selectively assigned.

CORRECTION OF HOME WORK

As hinted above, checking and correction of homework is absolutely essential. It is expected to be a routine activity with every teacher. If it remains unchecked, it does not fulfill its purpose. However, regular checking of homework is not an easy job. Thorough and regular checking is almost impossible, especially when a teacher is concerned with four or five different sections of students every day. In order to do some justice to this aspect of his job, he may have sample checking every day and thorough checking once in a while. He may also take the help of monitor and other brilliant students to do this duty on his behalf. Surprise checking may be another alternative. He may also introduce cross checking or checking by or checking exchange.

ASSIGNMENT

Assignment is a work allotment. It is the work assigned to the student. It may be done by the student at home or at school. There may be a pre-lesson or a post-lesson assignment. Assignment is a sort of undertaking or commitment on the part of the learner. He undertakes upon himself the responsibility of carrying out the work assigned. It may consist of solving a few mathematical problems, understanding a proposition or a group of propositions, solving riders based on a proposition, making many constructions of the same type, preparing illustrations for a topic, collecting mathematical data from the environment or literature, finding out applications of mathematical knowledge, tracing out historical background of a mathematical concept, formulating problem on a topic, carrying out a mathematical project, elucidation of an intricate mathematical concept, etc.

Assignment is a sort of self study which supplements class teaching. In the Dalton Plan, teaching is wholly based on the assignment system.

GUIDELINES FOR PREPARING THE ASSIGNMENT

1. Mere dictation of questions or problems is not an assignment. The teacher has to suggest alongwith it the book to be consulted and the references to be studied. Elaborate hints have to be given for the successful completion of the assignment.

2. The assignment should preferably arise out of the activities, needs and interests of the pupils.

3. It must motivate, clear up doubts or misunderstandings, and develop insight.

4. The task of assignment should be clear and to the point. It is not possible to achieve desired results by indefinite, wide-of-the mark, vague and lengthy assignments.

5. The assignment should be a cooperative activity in which the teacher and the pupils take an effective part.

6. Weekly assignments are preferable to daily or monthly ones.

CHARACTERISTICS OF ASSIGNMENT

Briefly speaking, the assignment must possess the following characteristics :

(a) Correlation with previous knowledge and experiences;
(b) Clarity and definiteness;
(c) Removal of the pupil's difficulties;
(d) Stimulating and directing the learning activity; and
(e) Recognition of individual differences.

PREPARATION AND EXECUTION OF THE ASSIGNMENT

Before preparing its written form the teacher must discuss and explain the work to be done. The assignment should be well timed so that it caters to the need of additional or supplementary work to be done by the students. The length of the assignments should be adjusted to the age and grade level of the pupils. The teacher may either write it on the blackboard, or dictate it or it may be given in the form of a cyclostyled or typed sheet. The hints, if any, should be given in a comprehensive way. It should not be allotted to the students as an unexpected and un-wanted burden. Assignment is to be considered as an undertaking for cooperative thinking and action. It has to be ensured that the students have accepted the activity. The difficulties must be anticipated and provision made for their removal. There should be advance-planning for the material and the literature to be used. While the assignment is being carried out, there should be proper supervision. The work should be checked, corrected and evaluated in the end. Suggestions may be offered for improvement. The assignment may be repeated after an interval, if it is found below the mark. A record of accomplished assignments by every student may kept in order to judge individual progress. The students may keep their completed assignments for future reference.

The giving and execution of an assignment may require a judicious choice of the source material, such as reference books, charts, pictures, maps and other illustrative aids. The teacher must familiarise himself with all the available teaching-learning material relevant to the assignment and thus direct his students accordingly. Reference books, encyclopaedias, good text-books, and other materials are always useful in stimulating and directing the students' assignment activity and independent study. A skillful use of source material always creates interest, defines the assignment work in hand, encourages self study, prevents failure and leads to success.

SUCCESS OF AN ASSIGNMENT

"The success and effectiveness of assignment depends upon the amount of independent work done by the pupils."

SELF STUDY

Self study is individual's own independent study. It happens when the individual studies and learns by himself. He coaches himself. He attempts and solves every problem without any outside help. Self study is a habit with the responsible and conscientious student. He waits for none to help or guide him. He is his own helper and guide. It is in every body's interest to depend upon self study. Ultimately every student whether brilliant or otherwise has to prepare for his examination by self study. He has understood things in the class, applied them under the teacher's supervision and discussed with any available helper in the home but when it comes to ultimate preparation and comprehension, it is better that he works all along. That is the stage when the student does not want any interference in his own work and wants to be left to himself. But self study is not to be employed only at the end, it should be a daily method for the learner throughout his education. Self study is the golden rule for self education. It need not remain confined to the limits of classroom work and the boundaries of the syllabus.

Self study can be made more systematic by giving regular homework, or assignments. It can also benefit by using various projects to be accomplished by the students as independently as possible. Library period is provided in the time-table in some schools with the purpose of giving them an opportunity of self study. Preparation for certain discussions, debates, declamations, symposia and other academic competitions would also require self study on the part of the students.

In mathematics, self study is essential to ensure regular progress. The work cannot be completed by the students without doing a major part of it as homework. The efficiency in solving problems is to be developed by the students own experience of solving them. One problem solved by self effort is worth more than ten problems solved by outside help. The experience of constructing figures is also to be obtained by the student by his own hands. The proofs of the propositions and more so of the riders have to be developed by the students largely by their own thinking and reasoning.

The following points may be kept in view to make self study more effective :

1. The habit of self study should be introduced and settled as early as possible in the young students.
2. The students should be made to understand that independent self study is most rewarding of all studies. It is the only method which will pay in long run.
3. Let the child work and learn by himself as far possible. The teacher should withhold his own self in order to allow the child's self to develop and grow by his self effort.
4. Some interesting problems be always presented to the students which may urge them to study and acquire information on their own.
5. The students should be intelligently directed to the sources of his self study. These sources should be of wider implications than the ordinary text-books.
6. They should also be guided to form the habit of note-taking in independent self study. They should also learn to make a note of their difficulties.
7. They may be advised to keep a record of the independent work done by them.

8. In case a student fails to show any output in his self study, the causes may be worked out He may be allowed to work with a more pushing partner.
9. The cooperation of the home should be obtained to ensure regularity and progress by a student in his self study.
10. The work accomplished by the student in self study should be checked, corrected and evaluated by the teacher.

GROUP WORK

Group work is a via media between class teaching and the individual's independent work. Since individual methods of work are almost an impossibility under the circumstances and class teaching does not fulfil the purposes of modern education, we have to strike a via media in the form of group work. Group work aims at securing the advantages of individualised instruction retaining at the same time the economic and socially advantageous way of teaching the students together. The class is a very large and unmanageable group whereas the group formed in this case will be small and manageable. The class is a heterogeneous group and suffers from sharp individual differences, whereas the group as such will be more homogeneous. In group-formation, the class is divided into smaller groups according to abilities and interests of the pupils. The unit becomes smaller and so greater individual attention can be paid. Over-individualism is also avoided. Children work under group feeling and there is more of cooperation, homogeneity and healthy competition.

In mathematics, there is ample scope of group work. In early stages, oral drill is conducted generally in groups. The mathematical tables are recited by the students in groups. The students may voluntarily engage in group work for doing their homework. They pool their resources and thus complete the assignment jointly which none of them may be capable to do single-handed.

In case a teacher teaches by activities, projects, assignment or practical work, the students find many opportunities of group work. The collection of some mathematical data from the field may also necessitate joint effort on the part of a group of students. Similarly visit to a place for having some mathematical benefit will be fruitful only if a number of students join to study different mathematical aspects of the institution or place visited. Preparation of a mathematical model or some other mathematical instrument or material will also require a group of students to work in cooperation. If the mathematics room is to be decorated and equipped for the subject, it will be a group work for the students.

The teacher should keep in mind the following points while using group work as a technique :

1. Group should be homogeneous as far as possible in the matter of intelligence and level of achievement.
2. The leaders, monitors, or dominating type students should not be allowed to monopolise the group work. Every member must get the due share of the activity.
3. The purpose of group work should be adequately clear to every member in order to ensure equal responsibility and effort.
4. The group should not be of unwieldy in size. The number of its members should commensurate with the work in hand. There should not be any rigidity about the membership of a group. The allotment of students to different groups should be a flexible affairs.
5. It does not imply that competition accompanies every group work. This role of competition need not be stretched too far. Group work should generate spirit of cooperation more than the spirit of competition.
6. The best age in which group can be most profitable is the gang-age from eight to twelve years.

REVIEW

This is one of the important and effective fixing devices. It is the mental process of 'going over' the material already learnt. It is to view again, or to repeat the work. The previous experiences are recalled for better memory and retention. It is meant for better understanding and command. It comes after initial learning. While drill is the repetition in original form of facts or skills already acquired, review is an intelligent and thoughtful repetition which introduces new elements and reorganisation of thought. The review recalls and renews and establishes new relationships. While drill is only a rehearsal, review is a reconstruction. It is the process of reworking of the learnt material. It is broader and more inclusive as compared to the drill. It involves not only reviewing, but also re-thinking, re-grouping, re-construction, re-arrangement and re-understanding. It takes place under the active supervision and guidance of the teacher.

PURPOSES OF REVIEW

It is employed for making up deficiencies and mastering of facts and skills, especially on the eve of examination. It enables the students to see the whole subject-matter in a new perspective. The subject-matter is initially taught in small units. The review unites these units into one. Sometimes an already acquired detailed explanation has to be converted into a good summary and review becomes a necessity here. The figurative, graphical and pictorial way of review may be effective. When the data is arranged in charts, graphs and tables etc., there takes place a good deal of reflective thinking. Reorganisation of the learnt matter may take place through oral discussion, summarisation, marginal notes, parapharasing, making graphs or maps, underlining important information, preparing a memorandum or scrap note-book etc. Where drill is likely to be a dull and mechanical repetition the review will remain interesing and thought provoking. In order to minimise forgetting of the acquired knowledge, a systematic and periodical review is often necessary.

CARRYING OUT REVIEWS

The review should not be used as an escape, when the teacher is unprepared with his day's lesson. It is not meant for keeping the students busy in such cases. Rather the occasion should be used to introduce new and interesting experiences in a varying manner. In continuation of the original work done by them on the problem, the students may be required during the review to prepare a summary, a report or outline of the work done, or a pictorial presentation.

SUGGESTIONS FOR THE SUCCESS OF REVIEW

1. Review is not just a repetition. It must involve new learning.
2. Review should provide a new organisation to the previous knowledge. It should bring about unity, coherence and continuity in the various units.
3. Review should focus on the main points rather than on the details.
4. Review should be focused on the points of weak assimilation in order to ensure complete group and gap-free retention.
5. Review may take the form of verification, checking or confirmation on of the learnt material.
6. Intelligent exercises or assignments given at the end of the chapters may be used for the purpose of review.
7. Graphical, pictorial, symbolic, etc., representation are very suitable techniques for the purpose of review.

Review is a part of the teaching process; it is a learning exercise leading to better reorganisation and retention. It must bestow new insights to the learner. In the case of review of practical activities, it must result into the perfection of skills.

SUPERVISED STUDY

It is self study on the part of the students but in the presence and under the direct supervision of the teacher. It may be conducted in the regular periods or in case of residential school in after-school hours. It can also be carried out in the reading room of the library where a teacher may be put on duty to watch and guide the students. Supervised study is a well-known feature of public schools where the resident students are to assemble at one place at night to study under the supervision of a teacher/teachers. Supervised study creates a formal and studious atmosphere for the self study of the students. It makes available on the spot help and guidance to the needy students. The teacher's presence makes the atmosphere more disciplined and congenial for hard work. The students are in the company of their classmates which makes mutual collaboration possible. It introduces regularity in work and ensures sustained progress. The mistakes of the students can be removed on the spot and their difficulties removed there and then. Under the arrangement of supervised study, every student has to devote the prescribed time compulsorily for homework or self study. None can evade or avoid. This is a good device for the lazy and shirkers.

In mathematics, the student can benefit a lot from supervised study. Even if he is a day scholar, he may be required to study on for some time after the school hour for supervised study. This period may be utilised for disposing of the home work. Sometimes the students cannot do their homework at home because they do not get any time for it or they do not get any guidance for the purpose from the parents or from anybody else. During the long vacations also, if the students could be called to school for a few days for supervised study, they may finish the homework allotted to them and thus may be left with the remaining holidays for full enjoyment. If the school provides some extra time for supervised study, the students may not have to carry away any burden of homework with them.

The following hints may be kept in mind for effective supervised study :

1. A congenial and comfortable atmosphere should be ensured in the place or room meant for it.
2. The atmosphere should not be very rigid and formal. There should be more of freedom and facility for mutual exchange and collaboration. The seating plan should also be informal and suit the convenience of the students.
3. The session for supervised study should not be too long. It should become tiresome and exhausting.
4. The teacher should actively supervise the students' work instead of sitting passively in a corner.

5. If need be, more than one teachers may be deputed to supervise the work of a batch of students in order to provide them help in all the various subjects and courses.
6. The teacher can take the help of monitor or brilliant students to share the work of supervision with him.
7. There should be a check on exemption or leave from revised study.
8. Supervised study is no substitute of self study, it should supplement the latter.

EXERCISES

1. Write short notes on the following techniques with special reference to the teaching of mathematics:

   (i) Drill work.
   (ii) Oral work.
   (iii) Home work.
   (iv) Group work.

2. What is the place and value of oral work in the teaching of mathematics ?

3. What do you understand by review ? What is its place in the teaching of mathematics ? How will you make review more effective ?

4. How will you employ the technique of assignment ? How is it different from home work ? How will you combine the two to draw more advantage for the students.

5. How is home work indispensable in the teaching of mathematics ? Give some hints to make it more effective as a technique ?

6. How do you distinguish between self study and supervised study ? What should be done to make the best of both in mathematics ?

7. Enlist and briefly discuss various techniques for the teaching of mathematics. How do they complement one another in the process of teaching and learning ?

CHAPTER FOURTEEN

MATHEMATICS TEACHER

As in the case of other teachers, many things are expected of the teacher of mathematics. His obligations not only are confined to the class-room but extend in many other directions also. It must not be forgotten, however, that his first and foremost obligation is to teach his subject effectively. Teaching mathematics is a task which, if sincerely undertaken, will challenge the best efforts of the teacher.

No teacher can do a thoroughly good job of teaching mathematics unless he is willing to make a careful analysis of his job and to be guided by that analysis in making his preparations and in conducting the work of the class.

There are two equally important aspects of any true profession viz., significant knowledge and effective technique. One cannot be efficiently professional if there is any serious weakness in either of the two.

The beginning teacher will need spend most of his time in improving his knowledge of teaching field and his techniques of teaching, and becoming familiar with the traditions and administrative policies of the school.

Gradually, however, his responsibilities outside the class-room multiply. These semi-professional duties will assume an importance second only to that of his class-room work. It will be appropriate here to discuss the nature, purpose and successful performance of these multifarious responsibilities.

THE RANGE OF TEACHER ACTIVITIES AND HIS PROFESSIONAL QUALITIES

His qualities and activities may be classified as follows :

1. Prerequisite qualifications.
2. Professional training.
3. Selective academic training.

4. Supervised practice teaching.
5. In-service training.
6. Professional activities.
7. School activities.
8. Mathematical organisations.
9. Departmental duties.
10. Administrative duties.
11. Community activities.

The scope of each of these activities varies considerably from school to school and community to community. But they are all significant and call for serious consideration by the teacher.

1. Prerequisite Qualifications

Mastery of the subject is an absolute essential for effective teaching. The teacher must possess a basic qualification in the subject. The level of his information should be much higher than that of the information he is expected to impart. Not only should he be the holder of a degree in the subject but his aptitude in the subject also must have been distinctly testified by his college teachers.

His personal characteristics are no less important. These include general appearance, health, physical defects, voice, poise, ability, energy, enthusiasm, cooperation, sense of humour, sense of justice, and willingness to seek the facts.

He should prepare himself professionally not only by a scholarship in mathematics, but by having cultural background courses in other fields as well, such as physical and biological sciences, social studies, language, crafts etc. This general education will give the teacher a broader outlook and better perspective which are essential for an efficient discharge of his duties.

Successful teaching experience prior to professional training is also a valuable asset. It will enable the person to acquire certain commendable characteristics such as promptness, adaptability, efficiency, the knack of arousing and maintaining interest, adequate command of instructional material and the ability to face the class with confidence.

2. Professional Training

Before going in for this training, he should make sure that he has enthusiastic interest in mathematics and in the profession he is going to adopt.

Many teaching failures in mathematics can be attributed primarily to a lack of understanding of the subject. Students soon become sceptical about the teacher's ability. The teacher should know not only mathematics but also other subjects of the curriculum. It is essential for the purpose of correlation.

He should possess a fair knowledge of the applications of mathematics. This is an extremely important factor in imparting to mathematics greater meaning and usefulness.

He should acquire a good knowledge of educational psychology to be familiar with the laws of learning. A knowledge of the various factors governing attention, interest, fatigue, perception, understanding etc., vitally affects his success in the class-room.

A course in the "Principles and Philosophy of Teaching" is also important for it will reveal to him the over-all picture of the theory and practice of teaching.

An elementary understanding of the history and the development of the educational system in the country is also necessary. Thus his mind will be oriented towards the present aims and purposes of education.

As a part of his preparatory training he is usually required to observe frequently the class-room work of experienced teachers in the practising school.

Teaching practice is also very important. It demonstrates the important factors of able class-room management and effective teaching techniques, which include many items of professional training. This practice enables the trainee to acquire practical experience and also gain self-confidence which can come only from accomplishment. During this trial period the pupil teacher should have the advice and assistance of expert teachers in the subject concerned.

3. SELECTIVE ACADEMIC TRAINING

This is to be acquired through college education in the subject. It is, in a way, a prerequisite for becoming a mathematics teacher. He should acquire a fair understanding of the broad field of mathematical analysis and reasoning.

Recent trends in the applications of mathematics demand that a course in Mathematical Statistics and Probability be incorporated in teacher training. Statistics has numerous applications in educational, economic, industrial, scientific, social and other problems. The would be teacher should have acquaintance with effective and reliable methods of collecting, organising, analysing and concluding about statistical data.

To enable him to understand applications of geometry and trignometry, courses in surveying and in the use of instruments including Mechanical Drawing are of considerable value.

A course in Consumer Mathematics will also provide a useful preparation for the profession.

He should first acquire a fair knowledge of the fields closely allied to mathematics and finally grasp all the useful techniques of teaching.

4. SUPERVISED PRACTICE OF TEACHING

The importance of this part of training has already been emphasised. It should be done under the direct supervision of experienced teachers. Its duration should be about one and a half months. The pupil teacher should get an opportunity of observing a few demonstration lessons by experienced teachers, and then he should be required to teach classes on those lines. He should also give a few discussion lessons. These lessons should be minutely observed and thoroughly discussed by a group consisting of his class-fellows and a few experts. The chief aim should be to offer suggestions for the improvement of student teaching. His lessons should be supervised and evaluated. Such evaluation should take into account aspects such as class preparation, teaching techniques, management and control, personal and professional qualities and achievements.

This practice teaching should truly mark the beginning of mature, reflective and analytic thought in examining and solving teaching difficulties.

5. IN-SERVICE TRAINING

It has been introduced recently in the profession. Its value and effectiveness in continuing the training of teachers to a higher plane of efficiency and competence are commonly recognised.

The extension services departments attached to different teacher training institutions are organising it. Seminars and workshops are occasionally arranged for the subject teacher, but much depends on the teacher's desire to learn and improve. The teachers should bring their problems and difficulties for free and frank discussion on such occasions. They should utilise the library service of the extension services departments. They should attend seminars and workshops with an open mind to learn. This is the training which should be availed of by them for keeping themselves up-to-date in respect of subject-matter and teaching techniques.

6. PROFESSIONAL ACTIVITIES

Each teacher should make definite provisions for his own professional growth and development. Many ways may be found for this purpose. The teacher should study mathematical journals and modern books of professional interest. Any facilities of in-service trainin should be availed of. Schools of outstanding merit may be visited to investigate the quality and nature of the work done. The teacher may do professional research work and write professional articles or books for publication.

The teacher may participate in providing for the programmes and discussions of professional organisations by becoming an office-bearer of an organisations or a committee member. He should at least join their deliberations and try to contribute his best views.

7. SCHOOL ACTIVITIES

The heads of the institutions are chiefly responsible for the actual carrying out and development of these activities by which the pupils can be offered the best opportunities for intellectual and physical development. Their effective formulation and execution, however, require the cooperation of the individual teachers. The teacher should enthusiastically participate in the following activities :

(i) Faculty meetings. (ii) Student projects for a special group or class. (iii) School clubs, social events, school publications. (vi) Excursions and other cooperative activities. (v) Committees for special school activities. (vi) Programme for a more effective cooperation of the mathematics department with other departments in school projects, educational fairs, tournaments, exhibitions, etc.

8. MATHEMATICAL ORGANISATIONS

Even a beginning in this field has not so far been made. Every teacher at his place has to think seriously about it. Such an organisation is long overdue. The subject teachers should organise themselves at district, zonal and state levels for mutual professional advantages. Such an organisation should provide a channel for exchange and sharing of experiences regarding different aspects of the teaching process. The objectives of these organisations should be academic and pedagogic rather than political.

9. DEPARTMENTAL DUTIES

For smooth functioning of a department each teacher must accept his share of responsibility. Similarly, the successful administrator has to secure active and willing cooperation of his staff through democratic procedure. A few common departmental duties are given here : (i) Participation in departmental meetings and discussions. (ii) Selection of text-books. (iii) Acting as sponsor of the mathematics club. (iv) Organisation of course syllabuses. (v) Selection and ordering of library books and periodicals. (vi) Selection and ordering of class-room equipment. (vii) Selection and ordering of Laboratory equipment. (viii) Preparation of departmental tests. (ix) Liaison committee, especially for cooperation with other departments. (x) Organisation of exhibitions, contests and assembly programmes. (xi) Administration and scoring of standardised tests. (xii) Taking charge of a remedial class in mathematics.

Many other such duties demand teacher's attention. These may be so distributed that a teacher usually need spend only two to three hours a week. Successful functioning of the mathematics department largely depends on successful discharge of these duties.

10. ADMINISTRATIVE DUTIES

Many administrative duties require the cooperation and assistance of the teachers. The average time to be devoted each day to them should not be more than a few minutes.

The teacher has to keep records of attendance and tardiness of pupils. These records serve teaching as well as administrative purposes. The records of an individual student achievements have to be kept. Summarised results have to be periodically reported to the administrative officer. Similarly the deficiencies of individual students have to be reported. The teacher may have to serve on administrative committees. He has to administer and score mental tests, and their results are to be used for diagnostic purposes and student adjustment purposes. He has to undertake guidance work. He has to select projects and help the students in executing them. Supervision of some of the rooms, hall and grounds can be another administrative duty for him.

11. COMMUNITY ACTIVITIES

In most communities, the school is the centre of educational life for parents as well as children.

Teachers should cooperate with parents to improve school and community relations and opportunities. There should be arranged meetings of social or civic groups, parents' association and old students' organisations.

Teachers should explain to parents and visitors the facilities offered by the school.

Teachers usually organise and direct local fairs or celebrations, lectures, variety programmes and other activities which are of interest to the community.

Teachers have to direct any service undertaken by their students for the uplift of the community. Teachers have often to work or direct work in an evening school for adults. They may be requested to help farmers in their business. Their advice may be sought by labour organisations.

The teacher is not an individual but an institution in himself. His work is great and his responsibilities are many. He is to be service incarnate.

EXERCISES

1. What type of personality should a mathematics teacher develop in himself ? Discuss and elucidate.

2. What are the qualities of a mathematics teacher ? What qualities should be possessed by him in addition to the common qualities of every teacher ?

3. Enlist and explain the qualities and characteristics of a teacher of mathematics ?

CHAPTER FIFTEEN

MEASUREMENT AND EVALUATION

THE EFFECTIVENESS OF instruction is usually determined by measuring achievements against objectives undertaken. An efficient programme of evaluation no longer comprises merely the effort to check the completed process but rather in the continual appraisal of the student's progress towards the attainment of pre-established aims. There is probably no more accurate barometer of the fundamental philosophy of any curriculum than a careful analysis of its evaluation programme.

PURPOSES OF EVALUATION

The main purposes of the evaluation programme may be listed as follows :

(i) To help provide more intelligent guidance in teaching and learning.

(ii) To develop more effective curricula and educative experiences.

(iii) To secure more intelligent and effective cooperation from parents and community.

(iv) To provide an adequate and objective basis for reporting progress.

The effective use of tests has now to form an integral part of the system of education. The evaluation programme has to be thoroughly comprehensive and balanced, and should not neglect any significant aspect of subject-matter. It must take into account all the important objectives that have been fixed. It has to integrate data of various types so that a composite picture of the individual is obtained.

Instruction is to be made life-centred, child-centred, activity-centred and what not, whereas examination is being allowed to remain mainly computation and cramming centred. It is wrong to emphasise certain values at the instruction stage of the process of education, and to emphasise entirely different values at the evaluation stage of the same process.

CRITERIA OF MEASURING TESTS

Certain criteria have been formulated which govern the construction, use and interpretation of all tests. The success of an evaluation programme ultimately depends on the satisfaction of these criteria. The most important of these criteria are :

(i) VALIDITY : It includes educational and statistical validity. In mathematics an achievement test designed for one level attainment is not usually meant for effective measurement at another level. Nor does a test designed to measure mechanical computational ability prove valid for measuring originality or ability to reason.

To be valid, a test should be worthwhile. It should measure abilities or traits for which it is intended. Similarly it should be valid for a given group of pupils.

(ii) RELIABILITY : It is determined by the consistency with which it measures that which it does measure. The behaviour of the examiner, the mental and physical condition of the student, and the conditions under which the test is given have a great influence upon the reliability of the result of any test. The ambiguity in the directions indicated in the test, lack of clearness in statement of questions, inadequate sampling of the matter to be tested, inefficient methods of scoring, and erroneous interpretations of test results also bring unreliability in a test. The more subjective the test is, the less reliable it is likely to be. To be highly reliable a test must be objective. The questions should be so stated that the answer to each one is definite and its correctness be not subject to personal opinion.

The validity and reliability of a test depend largely upon its statistical accuracy. The results of two or more tests intended to measure the same characteristics should exhibit a high degree of positive correlation.

(iii) ADMINISTRABILITY : It should be economical of the teacher's time. The amount of time required for the construction, administration and scoring of a test should not be excessive. The directions accompanying a test should be very clear and concise. Answer-keys for scoring should also be supplied.

(iv) STUDENT CONSCIOUSNESS : A test should not aim at confusing or puzzling the student. Reasonable tasks should be set for reasonable periods of time. In test items designed to measure understanding of a principle or ability to apply a principle, computation should be minimised.

(v) MOTIVATION FOR THE STUDENT : It should stimulate the best, genuine efforts of the student. The questions should discourage guessing and bluffing. The use of "catch" questions should be minimised. Occasional use of such questions might, however, be justified from the point of view of stimulation of accurate thinking. A test never be used as a means of punishment. It should always tend to create in the mind of the student the attitude of further study and effort. It should help create interest in the subject.

(vi) UTILITY : It is determined directly by the service which it renders. For teaching purposes, the results of a test are very helpful and useful in deciding the further line of action. Severe deficiencies in achievement should be taken to indicate the necessity of remedial teaching either for the entire class or for individual students. Minor deficiencies may be remedied by direct pupil effort. The analysis of the test results may also determine whether the teaching techniques employed are proving efficient for present purposes. This in turn helps to adjust standards and methods of teaching.

PROGNOSIS AND DIAGNOSIS

PROGNOSIS OR GUIDANCE

Prognostic tests are given to predict whether a student is likely to succeed in a particular field of work or not. In many aspects they closely resemble aptitude tests and certain types of intelligence tests. They are helpful in the enquiry into students' interests, aptitudes, work habits, and study skills.

DIAGNOSIS AND REMEDIAL TEACHING

The aim of diagnosis is to analyse the difficulties of a student in a particular phase of work. The aim is to reveal reliable information concerning his weaknesses in order to overcome them by concentrated action, and for detecting needs for remedial teaching. It may be used as an inventory test to find out how much a student knows about a given phase of the subject-matter before it is formally presented. Diagnostic tests are used along with personal interviews to discover and analyse pupil difficulties with a view to setting up specific remedial measures to correct errors and remove difficulties.

TWO MORE TYPES OF TEST

SURVEY TEST

It is usually broad in scope. It may cover the instructional material for one term of two or three months or for an entire year. Its chief purpose is to show the amount of student achievement and to indicate whether the student has sufficient basic preparations to proceed with advanced work of a similar nature. A good survey test can be used as a final test in a special topic or in a special subject.

ACHIEVEMENT TEST

It is used to measure the degree of mastery of skills, fundamental concepts, process and general knowledge of a subject, attained by the student. All educational tests are actually achievement tests used for particular purposes. The so called standardised achievement tests are somewhat broad in scope, and closely resemble the survey type. The informal tests given in class-room work are frequently referred to as achievement tests, regardless of the purposes for which they are used.

NEW TYPE TESTS

The programme of evaluation has to be designed in terms of the functional aims as well as the factual aims of mathematical instruction. The determination and perfection of techniques to be used in the evaluation of mathematical instruction is a definite responsibility of teachers of mathematics. Teachers should be extremely conscientious in the evaluation of student effort. Where other techniques don't help, judgment may be made on the bases of oral recitations in class, comparative class observations, personal interview, anecdotal record, and prolonged case study.

If prepared tests are to be used, the teacher will have to consider the comparative advantages and disadvantages of standardised tests and the tests made by him. Each possesses certain advantages over the other. The standardised tests are likely to be more reliable and valid, because they are generally constructed by persons of wide experience in both subject-matter and the techniques of evaluation. These tests are usually subject to a greater degree of objectivity in administering and in scoring. Another distinct advantage of the standardised test is that it diminishes the time which the teacher needs to devote to the details of testing programme.

On the other hand, it is probable that standard tests which are used year after year may exert some "backward influence" which might partially nullify their validity. It has been found that pupils generally do well on material that has appeared in a number of previous examinations. They have some scope of guess.

One of the major advantages of the teacher-made tests over the standard test is their flexibility and adaptability to local situations and repeated evaluations. These will include the material which the class has studied and no other material. Also these will lay adequate stress on the things actually emphasised before in the class-room. Other advantages of these tests lie in their relative inexpensiveness and inexhaustible availability.

Another question of special significance relates to the selection of the type of test, that is, "Where is an essay type test more suitable than a new type test, or conversely, where is a new type test more suitable than an essay type test ?" There are certain topics that seem to respond largely to the essay type or problem type test, e.g., geometrical propositions, construction of geometric figures, keeping of accounts, budgeting, taxes etc. The new type tests offer the opportunity of covering a wider range of material. They are likely to be more reliable and objective than essay type tests. Carefully constructed essay type tests also may have a high degree of validity. The objectivity of the new type tests makes them easy to score and removes any personal element from the scoring. They are, of course, no good tests of organising ability, and their construction requires much time, considerable experience and great care.

CLASSIFICATION OF NEW TYPE TESTS

To obviate the evils of essay type examinations, the new type tests have been devised. These constitute a combination of objective type tests, short answer type questions, oral tests and intelligent essay type questions. Objective tests are further classified into various categories. Classification is according to nature of content and the type of student response. Their names indicate the respective kinds of activities involved.

1. COMPLETION TESTS

In these tests are provided blanks that have to be filled in by the pupils. These should be carefully worded so that there should be in pupil's mind no doubt about the correct answer.

Examples : *Fill in the blanks* :

(i) The simplest way to express $\sqrt{{}^{2}y}$ is........

(ii) $a^5 \div a^7$—........

(iii) The square root of 1.44 is.........

(iv) A triangle can be constructed with two sides and..............
.........are given.

2. MULTIPLE CHOICE TESTS

The type of multiple choice test is composed of a number of items each of which presents two or more responses, one of which is correct. It is regarded as one of the best means of testing, and is definitely superior to the other types. It tests reasoning, reasoned understanding, sound judgment and discrimination.

(Tick the correct answer or answers).

Example : *They are always parallel straight lines* :

(i) Spokes of a cycle wheel.
(ii) Sides of quadrilateral.
(iii) The path of two wheels of a cart moving on zig-zag route.
(iv) Opposite edges of a table.
(v) Concentric circles.

(3) MATCHING TESTS

The type of matching tests consists of two columns where each item in the first column is to be paired with a word or phrase in the second column upon some basis suggested. There are many types of learning which involve the association of two things in the mind of the learner; such as terms and definitions, laws and illustrations, rules and examples, principles and their use, etc.

Example : *Match the results with the conditions* :

Result	Conditions
1. 45°	1. Right angle.
2. 180°	2. Reflex angle.
3. 360°	3. Straight angle.
4. 90°	4. Obtuse angle.
5. 135°	5. Angle at a point.
6. 450°	
7. 270°	

4. TRUE FALSE TESTS

Other similar forms are 'Yes', 'No'; 'Right', 'Wrong'; 'Correct' and 'Incorrect'. Tests of this type are very popular with teachers. Their merits are apparent in the case of construction, applicability to a wide range of subject-matter, objectivity of scoring and wide sampling of knowledge tested per unit of time.

Examples : *Answer 'yes' or 'No'*

(i) A book 8″×6″ can be contained into a box 12″×4″.
(ii) Two feet square is same as two square feet.
(iii) A triangle can be constructed when three angles are given.
(iv) Intersecting circles are concentric circles.
(v) Profit or loss per cent is always calculated on cost price.

5. SHORT-ANSWER TESTS

Here each item appears as a direct question. These belong to the category of simple-recall tests. The answer or response has to be recalled by the pupil from his past experience. It is differentiated from the essay examination primarily on the basis of length of response or answer required. Testing can be made more comprehensive with their help.

Examples : (i) Eight is what per cent of 64 ?

(ii) Which of the following data is inadequate, adequate and superfluous for constructing a triangle ?

(i) 2″ 70° (ii) 2″, 3″·5″, 4″, 60″ and (iii) 3″ 80°, 45°.

(iii) Two numbers differ by 7 The smaller is S. Express the larger.

(iv) What more do you need to construct a triangle when the following data are given ?

(i) Two sides, (ii) Altitude, (iii) Two angles, (v) Three angles.

(v) Find the side of an equilateral △, if the perimeter is 36 inches.

Correct the statements :

(vi) A triangle can be constructed when three sides and an angle are given.

(vii) A triangle can be constructed when altitude and base are given.

ORAL TESTS

Oral questioning is equally important for purposes of instruction and measurement. It can be used with an individual pupil to probe his reasons for his particular way of answering certain questions in other tests. That will help in determining the causes of error. It will further help in knowing how well an individual has integrated his knowledge, can apply it to new situations, sees its application, and can hear, understand, think, and express with alertness.

There is no need of giving separate examples to illustrate these tests. Almost all the questions suggested for objective and short-answer tests can be used for oral testing with slight modifications here or there.

But the use of oral tests is very time consuming and highly subjective. It has also relatively little utility in the class-room especially as a basis for determining pupils' marks in a course. It is not equally fair and just to all pupils. Some shy students may not do so well as they are capable of doing. It does not test extensively and permits interference and favouritism, whether intentional or unintentional.

In spite of all these drawbacks, oral testing is an absulute necessity in lower classes and is a significant part of even the highest examination.

INTELLIGENT ESSAY-TYPE QUESTIONS

One must note certain disadvantages of objective tests, which can be covered up only through good essay-type testing.

DISADVANTAGES OF OBJECTIVE TYPE TESTS

Perhaps the most serious disadvantage is that extreme emphasis is laid on factual knowledge, and practically no importance is attached to processes, methods of procedure, or originality of thought and expression. The subject of mathematics is a mode of thought rather than a group of isolated memory facts.

Another disadvantage is their failure to provide for systematic organisation and expression of thought.

In objective tests, answers may frequently be guessed from suggestions given in the wording of the questions or from associated ideas. Even when this guess element has been reduced to minimum, guessed answers definitely affect the score.

It is very difficult to prepare good objective type tests. To select questions judiciously, word them properly, and apportion them statistically requires considerable time, effort and experience.

The tests may also prove very expensive. When required in large quantities, both for preparation and for use, printed forms of objective tests may prove to be quite a costly affair.

Obviously the objective, the short answer, the oral and the essay tests serve different purposes and should be used accordingly.

The trouble with the essay-type test is that, although it has been in existence for hundreds of years, the amount of research devoted to it is much less than that to the objective test. Further, all the research relating to essay type has been of a negative kind. It is just as important to know where to use the essay type examination as it is to know how to use it.

Examples of essay type tests : —(i) Define a straight line and from its definition prove that any two sides of a traingle taken together are greater than the third.

(ii) A vendor purchases 45 kilograms of tomatoes at the rate of 5 kgms. for a rupee. He sells them at the rate of 30 paise a kgm. till noon time, and the rest at the rate of 18 paise a kgm. On the whole he earned Rs 5.00. How many kgms did he sell at different rates ?

EXERCISES

1. What are the major shortcomings of the evaluation programme in mathematics ? How would you try to remove them in your own teaching programme ?

2. What are the different types of tests suitable for testing in mathematics ? Illustrate them with examples ?

3. Discuss the place of new type tests, oral tests, short answer type tests and essay type tests in the testing programme of mathematics ?

4. Write an essay on the need of improvement in the technique of evaluation in mathematics.

CHAPTER SIXTEEN

THE TEACHING OF ARITHMETIC

WHAT IS ARITHMETIC ?

ARITHMETIC IS THE science of number and the art of computation. Historically, arithmetic developed out of a need for a system of counting. It is considered to be essential for efficient and successful living. The need of a good command of arithmetic by a house-wife, by a modern farmer, by a successful merchant, by a skilled worker, and by a progressive professional man, is too obvious to need any discussion. Also its utilitarian, cultural and disciplinary values are too obvious to need any argument at this stage. The teaching of arithmetic has to fulfil two major responsibilities : (1) the inculcation of an appreciative understanding of our number system and an intelligent proficiency in its fundamental processes ; (2) the socialisation of number experiences.

AIMS OF TEACHING ARITHMETIC

The following are some of the aims of teaching arithmetic :—

1. To teach the learner the mathematical type of thought to understand statements, to analyse them and to arrive at right conclusions.
2. To arouse the child's interest in the quantitative side of the world around him and its use as a simple tool in business.
3. To give accuracy and facility in simple computations of the fundamental processes.
4. To impart a working knowledge of practical arithmetical applications which are useful in life.
5. To prepare the way for higher mathematics.

METHODS OF TEACHING

The methods already discussed have to be applied for the teaching of arithmetic according to the type and nature of different topics. The emphasis will be on the use of the combination of Inductive-Deductive. Other methods will also be adopted when necessary and convenient.

The teaching of different topics of arithmetic as discussed below will give a better idea of how and by what methods these have to be taught.

NUMERATION AND NOTATION

Formal lessons in Number are of no use to the learner unless he has had a sound and varied experience of number, and until he has acquired a sufficient vocabulary in the language of number.

NUMBER SENSE

It is generally believed that the 'Number Sense' is something innate with the human being. A child of two years is able to distinguish between big and small and more and less things, especially when they are eatables. Since the sense of number is something natural, the primary school teacher has to develop it and not to create it. Even some of the animals seem to possess the 'number sense'. A chimpanzee knows that five bananas are more than four bananas. There is an interesting experiment about the crow. When the eggs from its nest are removed one by one, it does not seem to notice the removal, unless the number is reduced to three. Then it starts flying in search of the lost eggs. That is the crow has a sense to distinguish between three and more eggs.

NUMBER NAMES OR NUMERATION

While playing from morn to eve the child teaches himself the beginnings of number, more effectively if his environment is sufficiently varied to arouse his continual curiosity. A child knows that there are more members in his house than in the neighbour's; that his friend has more brothers and sisters than he has; that he is four years old; that dogs and horses have four legs, and that a man has two; that he has five fingers on each hand ; that he has to go to bed when the clock strikes nine ; and that he sits fourth in the first row in his nursery class.

A child is continually meeting with these and similar number facts and names both at home, in the environments and at school. In the class-room his experiences have to be made real and extensive. The language of number will become real if the child gets an opportunity to work at real games and toys. There are stories of 'Twelve Weavers, and the Peasant', 'The Hare and the Tortoise,' 'Three Blind Mice'. Many interesting games can be devised. Birthdays are a never-ending source of interest, and much can be made of them, for each boy on his birthday must know how old he is.

There should be many objects and toys and boxes containing scores of coloured balls, beads, tablets and sticks. Let the child play by all means, but let it be real play. The teacher should try to lend reality to all the activities of the child, and also must take care that the child knows the meaning of the terms that he uses.

Children love counting. It is a fascinating recitation to them. Even at the roll call they may sometimes be asked to count the children present in the room. A normal child, if he is allowed to throw balls into a basket and count them simultaneously can acquire acquaintance with and speed in number names.

If natural opportunities for counting and matching groups, cups with saucers, chairs with tables and the number of students present are in sufficient variety, many children will not need any special counting apparatus. Abacus counters, or some other tallying apparatus, should, however, be part of the equipment of every class-room. In a basic school, the counting of rounds, slivers, spades and khurpas can be good alternatives. Prolonged use of special apparatus, should generally be avoided. Any special apparatus or type of pattern, however attractive, should not be allowed to become dominant. It is wise and expedient to allow variety and experiment. Self-made or self-acquired patterns should also be encouraged.

NOTATION

Closely related to the reading of number symbols is the writing of numbers. The two operations may be considered as part of the same job. Practice in reading the symbols is helpful for instruction in the writing of numbers. Observing 'Number picture-cards' and moving a finger in grooved symbols, can be helpful exercises.

Children should not be asked to read and write large numbers which are beyond their comprehension. There is a sound principle to be kept in mind, "Educate the children with numbers in children's size." The writing of those numbers will be interesting for him which arise from different situations of his life, such as his roll number and his age.

Notation involves three operations; the student has to connect these three :

1. He is to recognise the figure and form the idea of its name.
2. He is to associate the symbol with the name.
3. He is to write it down.

Just as the child counts beads, he must be able to recognise the difference between different numbers of the beads. Alongwith the recognition, there will be assigned by him a name, one, two, three, four etc., to the groups. This is the formation of the idea of number name. For depicting them in symbols, he will write down, one two, three, four and so on.

Even from the environments, the child learns to associate the figure shapes with the number names. Figures occur on house doors, shop prices, omnibuses, clocks, calendars or the pages of a book. If the child is in the company of a sympathetic adult, he may learn that the number written on the house is 8, or that the car passing by them bears number 9.

While children are learning to write numbers, the teacher should be very careful about the manner they form the figures. If left to themselves they are liable to invent peculiar ways to form them and thus fail to employ economical movements. The teacher must, therefore, closely supervise their work at this stage.

By the time the pupils have learnt the notation for the first nine numbers, they may have advanced in numeration upto a hundred. The teacher will then tell them that there is no special symbol for the subsequent numbers and that they have to make use of the nine symbols to denote all other numbers. This introduction should also be based on the observation of written numbers having two digits. When it comes to ten, the symbol of zero has also to be given. It is not advisable to begin with zero.

OUR NUMBER SYSTEM

The expression "number system" is frequently used with only a vague idea of what is implied by the term 'system.' The basic idea of the number system is its four outstanding characteristics.

The Hindu-Arabic is the system used for nearly all purposes by Indians. The importance of this system of notation needs no emphasis. For a few purposes is used another well-known system, the Roman. But the superiority of the Hindu-Arabic system over the Roman system can be easily shown by comparing and contrasting the two. There is a sort of basic consistency in the Hindu-Arabic system which is missing in the Roman system. The same principle is used in expressing units, tens, hundreds and other higher numbers in the Hindu-Arabic system. The value of a numeral in this system is determined by its position. For example, the numeral 5 represents five, ten times five, and hundred times five in the numbers 5,50,500 respectively No such convenient and regular multiplication of value is caused by moving a Roman numeral from one position to another in a number; in the Roman system, the value of each numeral is always independent of its position. It is the principle of multiplying the value of a numeral by changing its position in a number that gives to the Hindu-Arabic system its matchless capacity for expressing in simple form any number, no matter how large or how small it may be.

ITS CHARACTERISTICS

1. The Hindu-Arabic numerals, if used in the ordinal sense, always have the same order; 4 always comes after 3, 6 after 5 and so on. This order holds good regardless of the size of the qualities (units, tens, hundreds. . .) referred to. Since the number namesalways have the same order, learning the relation (say, of five thousand to six thousand) presents little difficulty.

2. The first nine numerals represent quantities thought of as groups of ones, or units. But ten is thought of as a single entity or collection, like a ten-rupee note or ten fingers of both hands. This fact is shown even in the name "ten", where the familiar terminology of 1 is implied. The use of collections of ten makes the Hindu-Arabic number system a decimal system universally acceptable. It has a base of ten. When the idea of collection is properly utilised and taught, 11 is not eleven separate entities; it is two entities consisting of a collection of one ten and one unit. Thirty is three entities, or three collections; it is three tens. The idea of "collection" makes the addition of three thousand and four thousand as easy as the addition of three and four.

3. The names for the tens consist of terms that are closely related to their meaning. Twenty can be easily related to two with tens; thirty is three 10's; forty is four 10's. Thus, in dealing with the tens (twenty, thirty,......) a person does not have to learn entirely new names with a special place in the series. For the most part, the names in the tens series are much like those of the first nine numbers. The order of the tens is exactly the same as that of the numbers from 1 to 9. In learning to count upto one hundred, the similarity of the ten's names to the one's names makes the task of the learner easy.

4. The use of place value simplifies the writing of numbers. The value of a digit is dependent on its position in a number. As an illustration, consider the numbers, 5, 53 and 580. In each number the numeral 5 has a different value because of its different position. This principle combined with the use of a place-holder (zero) makes possible the writing of any number using only ten different symbols. The economy possible in this system can be clearly seen by comparing it with the Roman system; e.g., by writing the year nineteen hundred sixty three in the two systems : MCMLXIII and 1963.

THE ROLE OF ZERO

The Hindu-Arabic system of notation with its positional value is made possible by the use of a place-holder, namely zero. There is no definite information available regarding the discovery of 'zero', or even the people who first used it. It was once thought that the Arabians discovered it. Later on, the Hindus were given credit for it. Now it is known that the Babylonians had the idea and that they probably borrowed it from others.

Without doubt, the discovery of zero will always rank as one of the major steps in man's ascent to his present state of civilisation.

The idea of zero is so simple that many teachers have never given a thought to its significance. The important point to consider, and to teach, is that zero is used in notation to hold a place without any numerical value. It is mainly used to fill vacuums in notation. It is not a number in the sense that it represents a quantity. The symbol 1 in 10 is not different from the 1 representing unity, excepting that it is in the second position from the right. There is zero or nothing in the first place. Children's understanding of 10 will be enhanced if teachers frequently refer to it as one ten. It really is one ten ; similarly 20 is two tens.

The "place-holder" function of zero is its most important use and should receive major attention.

In current practice, the symbol 0 is read not only as 'zero' but as "nought" and "cipher" also. For example in the giving of telephone numbers, the use of "nought" is more popular and useful. In addition to these names, the symbol 0 is read in another way also when it appears as an integral part of a number. For example, in reading each of the numbers 50,505 and 220,354, the zero affects the reading, but is not sounded as "zero" or "nought".

Sometimes "zero" remains silent in reading such as in 505 the reading is "five hundred and five." Some teachers advocate that the use of "and" in such cases is superfluous. It should be straightaway read as 'five hundred five". Whatever be the practice, the zero remains unspoken. There can be a third way of reading it i.e., "five nought five", which also conveys the sense correctly.

THE TEACHING OF FOUR SIMPLE RULES

For some years every young learner is concerned mainly with the so-called 'four simple rules'. Proficiency in these processes is very important. The student has to depend on these at all the stages of learning mathematics. These are foundational in nature. The entire structure of the subject has to be built on this foundation. It is customary and natural to enable the child to acquire speed and accuracy in these in the very beginning.

The most important thing in teaching these rules is that the preliminary experiences should be given in an interconnected form with the help of concrete material. Ultimately, for the purpose of practice, their teaching will take abstract form. Their operations have to be taught side by side as far as possible. The teacher must impress upon the students the educational values and necessity of the avoidance of errors in these operations and of their careful execution.

He must not allow any such errors to persist, otherwise the learner's later performances are bound to be defective.

First We Take Up Addition :

The ideas of addition and subtraction arise at the same time, the one being the inverse of the other. Subtraction is the process of finding the number, which added to the less will make the greater. The idea of 6+4 makes 10, involves the idea that 10 is 4 more than 6, or 6 more than 4.

Before studying the teaching of any area of arithmetic, it is well to consider some of the fundamental characteristics of the area. The student may ask, "What is addition ?" and "Why do we add ?" The teacher may have to depend on some verbal or concrete problem situation to explain it. "There were six children on the bench and four on the ground. How many children were there in all ?" He goes on, "through the process of addition, the two groups of children can be combined and treated as a single group of children. This combination of two or more groups or quantities into a single group is known as addition." The student asks, "Why add ?" The answer is, "We add to simplify a quantitative situation. More numbers can be expressed into a single one."

The instruction in Addition should begin as soon as the child begins to count rationally. Just as counting is done orally, similarly preliminary addition should also be done orally. Emphasis should be laid on the acquaintance with and the mastery of the basic addition facts.

There are in all one hundred addition facts.

$$\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}$$

$$\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array} \text{ and so on.}$$

Some have reduced the number to 81 by eliminating the additions involving zero.

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}$$

$$\begin{array}{ccccccccc}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array} \text{ and so on.}$$

These facts can be presented through charts. After adequate continuous practice in them, the following order can be followed :—First concrete numbers then abstract numbers, first smaller numbers and then bigger numbers.

In the first instance, addition of numbers of one digit is to be taken :

5 beads+3 beads.=8 beads.
and then 5+3=8

Then addition with simple compound numbers. Ask them to add 3 sticks and 2 beads, 4 sticks and 3 beads. Then let them add Rs 3,4 Paise and Rs 5,8 Paise.

Then addition with numbers of two digits, so that no carrying is involved. Add 34 beads and 45 beads. Add 43 and 56 and so on.

The addition with more difficult compound numbers like add Rs 45, 12 Paise and Rs 52, 36 Paise.

Then addition with numbers of three and four digits.

After some exercises, when the teacher finds that the pupils can add numbers involving no carrying, he may give a problem, the solution of which involves carrying. The help of the addition box may be taken. It may be a simple box having eight compartments. It should be impressed upon them that not more than nine things can be kept in each of the compartments of the box. Ten will be expressed by having one bead in the 'Tens' compartment and none in the 'Units' compartment. Supposing they have to add 8 units and 7 units. Doing it practically they will get 15 pieces in the 'Units' compartment. But this will be a wrong expression. To make it correct they will have to carry one piece to the 'Tens' compartment, keep five pieces in the 'Units' compartment and throw away the rest. Thus a continuous practice with this box will remove the difficulty of carrying.

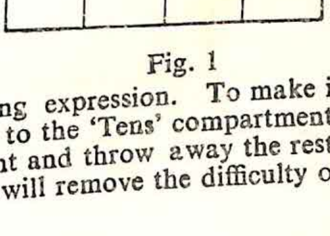


Fig. 1

SUBTRACTION

Subtraction is needed to find the difference between two numbers. The need to find the difference may arise in a variety of ways which may be classified under two heads. One may have to find the remainder, or may have to make up deficiency. In one case something is taken away, while in the other something is added. Therefore subtraction can be taught on the same lines as addition. In early stages it must be taught as such.

As in the case of addition, there are alternate one hundred basic subtraction facts.

$$\begin{array}{cccccccccc} 0&1&2&3&4&5&6&7&8&9\\ 0&0&0&0&0&0&0&0&0&0\\ \hline 0&1&2&3&4&5&6&7&8&9\\ 1&2&3&4&5&6&7&8&9&10\\ 1&1&1&1&1&1&1&1&1&1\\ \hline 0&1&2&3&4&5&6&7&8&9 \end{array}$$ and so on.

And just as in the case of addition these facts were reduced to 81 so in the case of subtraction these have been reduced to 81, by eliminating the subtraction facts involving zero.

$$\begin{array}{ccccccccc} 2&3&4&5&6&7&8&9&10\\ 1&1&1&1&1&1&1&1&1\\ \hline 1&2&3&4&5&6&7&8&9 \end{array}$$ and so on.

Sufficient oral practice in these facts should be given to the beginner. The order of the difficulty, given in the case of addition problems should be followed in this topic as well.

Moreover, there are three methods of subtraction.

THE DECOMPOSITION METHOD

In this method, the bigger number is broken up into a more convenient form. This is the most popular method with the teachers today. He explains as follows :—

Instead of $\begin{array}{cccc} \text{Th}&\text{H}&\text{T}&\text{U}\\ 3&4&5&6\\ 2&9&7&8\\ \hline \end{array}$ we use $\begin{array}{cccc} \text{Th}&\text{H}&\text{T}&\text{U}\\ 2&13&14&16\\ 2&9&7&8\\ \hline \end{array}$

Eight cannot be subtracted from six, therefore we have to borrow ten from the tens' place to make it sixteen and so on." It is clear that once the decomposition is performed, it is immaterial whether work at the sum is begun at the right hand or at the left. In actual practice the decomposition is performed mentally.

THE METHOD OF EQUAL ADDITIONS

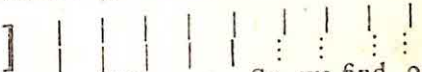
This method is based upon the simple axiom that if equals be added to unequals, the original difference remains constant, e.g., $8-4=18-14=118-114$ etc. The ages of children provide an interesting instance, because as long as both are alive the difference remains constant, whatever their individual ages. This method is applied as follows :—

$$\begin{array}{lr} & 3465=3000+\ \ 400+\ \ 60+\ \ 5 \\ & 1786=1000+\ \ 700+\ \ 80+\ \ 6 \\ \text{The upper row is written as :} & 3000+1400+160+15 \\ & 2000+\ \ 800+\ \ 90+\ \ 6 \\ \text{Now subtraction is performed} & 1000+\ \ 600+\ \ 70+\ \ 9 \end{array}$$

The answer is 1679. This method is undoubtedly a lengthy method.

METHOD OF COMPLEMENTARY ADDITIONS

Here a subtraction problem is viewed as an addition problem. The teacher says, "The difference between two numbers is found by adding on to the smaller, until we make up the larger.

Thus in the case of 9—5, we make up the deficiency of the  smaller number and find out 4 as the difference. So we find out what must be added to the smaller so that is becomes equal to the larger." It is this later aspect of adding on the complement or complementary addition which should be the natural and normal process of working a subtraction sum. Some say that this method should be made universal. It is also known as the method of interrogative addition. The word subtraction need not be mentioned at all, because the whole process may be taught as addition in a new form. Thus the above problem may be arranged as :

1 7 8 6
× × × ×
3 4 6 5

in the form of an addition sum where the answer is given and the missing line has to be found.

Here are a few more hints for the teaching of subtraction :—

1. Formal methods should not be introduced too early in the child's life. Develop considerable proficiency in oral work, before any routine method is adopted.
2. It is essential that all the teachers in the same school agree to follow the same method. It is unfair to the students to be taught one method in one class and another method in the next class.
3. Develop the habit of invariably checking any subtraction sum by the corresponding addition.
4. Where a number of zeros occur, the work is lessened by first subtracting one from each quantity such as :

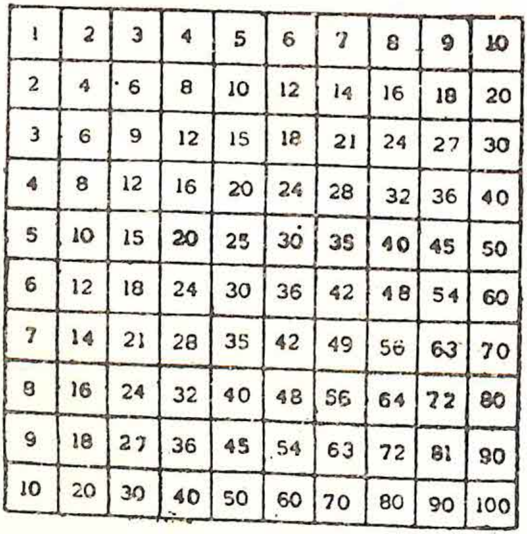
$$\begin{array}{lr} 1000 \text{ is much easier for a child when worked as } & 999 \\ 658 & 657 \end{array}$$

MULTIPLICATION

For the beginner, multiplication is best regarded as a special case of addition in which all the numbers to be added are the same. The first experiences of multiplication should come or be given through practical activities using concrete material. The construction of multiplication tables by the method of equal addition and by using concrete material, is one of the first tasks of the learner. For example :—

$$\begin{array}{ccr} 1 & 3 & 3 \\ & & \underline{3} \\ 2 & 3 & 6 \\ & & \underline{3} \\ 3 & 3 & 9 \\ & & \underline{3} \\ 4 & 3 & 12 \\ & & \underline{3} \\ 5 & 3 & 15 \end{array} \text{ and so on.}$$

Why do we multiply ? is the question which is likely to arise in the very beginning. The teacher will again have to depend on a problem situation. He asks. "Suppose the price of a book is five



1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Fig. 2

rupees, what will be the cost of four books ? Or one kilo of a thing costs three rupees, what will be the cost of 6 kilos ? The students have to be told like this—"So we multiply to simplify such situations. Multiplication is the process which enables us to deal with many things, when any particular character of one thing is known."

But the mere construction of tables is only preliminary work. The exceedingly important, and more tiresome work still remains, viz., the tables when constructed must be memorised. The teacher should not be afraid of the drudgery of repeating tables. Proficiency in multiplication depends upon a ready knowledge of the tables need. The table not be mechanically taught in consecutive order from the table of "twos" to a table of "twelves." The table of "fives" is easier than the table of "fours" and the table of "elevens" is easier than the table of "nines" or "eights," while the table of "tens" is the easiest of all. Let the following order of difficulty be adopted and tried by the teacher : 1. Tens. 2. Twos. 3 Fives 4. Fours. 5. Elevens. 6. Eights. 7, Threes. 8. Sixes. 9. Nines. 10. Twelves. 11. Sevens.

It is customary these days to summarise these tables in the form of the number squares. This table is more convenient and can be both extended and reduced. It can be written on one chart and hung on the wall of the class-room to be observed by the students throughout the period of learning the tables. This square upto ten is given on page 220.

Short multiplications seldom give much trouble. All that is necessary is a ready knowledge of the tables, because the multiplier is within the digits of the tables. In long multiplication, the most important thing is to multiply first by the right-hand figure of the multiplier and so on. Care should be taken that the right-hand figure in each product falls below the figure used as a multiplier.

Multiplication is not to be regarded merely as a mechanical process, for it constantly provides scope for the use of intelligence and also a vertical paradise of "short cuts". There are some cases where multiplication is simplified by subtraction, as in the case of multiplication with 9,99,990 etc., $9=10-1$, $99=100-1$, $990=1000-10$. There are other cases where the labour of multiplication can be lessened by a careful examination of the digits of the multiplier, as in the case of 357, it is known that $35=5\times7$. Sometimes, multiplication can be avoided by simple division as $5=\frac{10}{2}$, $25=\frac{100}{2}$, $3\frac{1}{8}=\frac{10}{8}$ etc.

DIVISION

This process is usually placed last of the four rules in an ordinary text-book. This is undoubtedly its logical position, because a good knowledge of addition, subtraction and multiplication is necessary for the understanding of the theory underlying this process. In the first place, division may be regarded as shortened subtraction, just as multiplication has been considered as shortened addition. Thus the question involved in 36÷4=? may be worded as, "How many times can 4 be subtracted from 36 ?" Division can equally well be defined as shortened addition, for the above question may be worded as, "How many 4's must we add together to make 36 ?" It is the reverse of multiplication; therefore it can equally well be viewed as interrogative multiplication. The same question can be worded as, "By what must I multiply 4 to get the answer 36 ?"

To answer the question, "What is division ? Why do we divide ?", the teacher may again resort to some problem situation. He says, suppose there are a dozen biscuits, and four brothers and sisters have to share them equally. What will be the share of each ? Or, ten oranges cost Rs 1-50 paise. What is the cost of one orange ? Division is sharing or finding the value of one when that of many things are given."

So long as divisors do not exceed the limits of the multiplication table, it is customary to work by the process called short division. As in the case of multiplication, success here also depends upon a ready knowledge of tables. Hence constant oral practice on these tables is necessary. He should tell the students that in multiplication the process is performed from the right hand side to the left hand side whereas the process of division is carried out from left to right. The carrying over is also done in the same respective directions.

DIVISION BY FACTORS

This process follows naturally from 'Short division'. This divisor is broken up into simple and convenient factors. But finding the remainder has always been recognised as the greatest difficulty in this process.

PROBLEM

Divide 4849 by 42 by the method of division by factors ?

Divisor 42 can be broken up into two factors—6 and 7.

$$\begin{array}{r|l} 6 & \underline{64849} \\ 7 & \underline{\quad 808 \quad} \end{array} \quad \text{groups of six} + 1 \text{ unit.}$$

$$115 \text{ groups of } 42 + 3 \text{ groups of } 6 + 1 \text{ unit.}$$

Therefore Quotient $= 115$

and Remainder $= 3 \times 6 + 1 = 19$

LONG DIVISION

It is certainly the most difficult of the fundamental processes to teach, as well as the most tedious to perform. In division, we have four distinct quantities, (*a*) Divisor (*b*) Dividend (*c*) Quotient (*d*) Remainder. The straightforward process is given (*a*) and (*b*), find (*c*) and (*d*). It is essential that the child should know the connections between these four quantities. He should further learn that every division sum can be checked by multiplication according to the equation : Dividend = (Divisor × Quotient) + Remainder. He should learn that divisor and quotient are interchangeable (so long as the remainder is less than either). Hence a division sum can often be checked by a second division, this time using the quotient as divisor.

LCM AND GCM

Both Greatest Common Measure and Least Common Multiple play an important part in ordinary fractional manipulations. The point to be stressed is : these two processes are subsidiary processes and are not to be taught as if they were important ends in themselves. They are processes having but few applications in ordinary life. To persist in setting examples such as finding the GCM of 45676 and 34592, is not justifiable.

THE TEACHING OF GCM

The following stages are suggested :

(1) Oral examples in numbers and quantities, such as, "What is the greatest number common to 6 and 8, to 6, 9, 12 to 14, 21, 35 etc ?"

(2) More difficult examples requiring the use of written work but which can be solved by finding the prime factors of the numbers, e.g.,

"Find the GCM of 72 and 162; of 65, 91 and 69, etc."

The method of finding GCM by factorisation is the most sensible method for children. All that is necessary is ability to resolve each number readily and accurately into its prime factors. The required greatest measure can be easily seen from an examination of these prime factors.

Example : Find the GCM of 264, 504, 624.

Factorising—$264 = 2\times2\times2\times3\times11$

$504 = 2\times2\times2\times3\times3\times7$

$624 = 2\times2\times2\times2\times3\times13$

An examination shows that the GCM $= 2\times2\times2\times3 = 24$

THE FORMAL METHOD FOR GCM

Let this method be applied to the example—Find the GCM of 450 and 125. The work is arranged as—

```
125)450(3
    375
    ---
     75)125(1
         75
         --
         50)75(1
            50
            --
            25)50(2
               50
               --
                ×
```

This method can be explained by the Method of Lengths—Take two strips of paper, depicting lengths 450 and 125 respectively.

Mark off the shorter strip as many times as possible from the longer strip, as in the diagram. The remainder is seen to be 75. Next mark off this remainder as many times as possible from the shorter strip, as shown in the second diagram. The second remainder is seen to be 50. Now mark off the first remainder by the strip of the second remainder, as shown in the third diagram. The third remainder is seen to be 25 which is exactly contained in the second remainder and is, therefore, the factor or measure required.

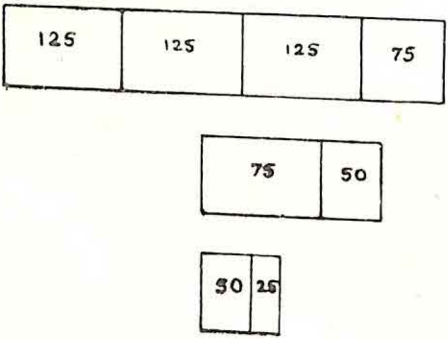


Fig. 3

This method applies directly only to two numbers or quantities. For more than two numbers the process must be repeated.

APPLICATION OF GCM

The most general application of GCM is in the cancelling of fractions or in their reduction to lowest terms. Examples in areas involving the finding of the greatest possible square tile, block or paving stone are frequent in arithmetic books.

METHOD OF FINDING LCM

The idea underlying this term is akin and complementary to that of GCM, and the methods of finding them are usually taught concurrently.

As with GCM, so in LCM, the best beginning can be made with oral work with simple examples.

Examples : What is the smallest number which will contain both 4 and 5 an exact number of times ? (20). What other numbers will also contain 4 and 5 exactly ? (40, 60, etc.).

These examples should be varied, as much as possible, using the simplest convenient numbers.

The next stage should concern examples which can be easily factorised, as—Find the LCM of 12, 16, 18 and 27.

Solution

$$\begin{aligned} \text{Factorising—}12&=2\times2\times3\\ 16&=2\times2\times2\times2\\ 18&=2\times3\times3\\ 27&=3\times3\times3 \end{aligned}$$

Taking the factors in order, the LCM is obtained as

$$2\times2\times2\times2\times3\times3\times3=432$$

The underlying theory is simple and straightforward. The maximum occurrence of 2 is four and that of 3 is three.

THE FORMAL METHOD

The above example, worked formally, would appear as follows :

$$\begin{array}{r|rrrr} 2 & 12, & 16, & 18, & 27\\ \cline{2-5} 2 & 6, & 8, & 9, & 27\\ \cline{2-5} 2 & 3, & 4, & 9, & 27\\ \cline{2-5} 3 & 3, & 2, & 9, & 27\\ \cline{2-5} 3 & 1, & 2, & 3, & 9\\ \cline{2-5} & 1, & 2, & 1, & 3 \end{array}$$

Ans. $2\times2\times2\times2\times3\times3\times3=432$

The method is clearly one in which the prime factors necessary for the answer are found by successive division. The choice of divisors should not be haphazard, and as a precaution against error beginners should be advised to divide by prime factors only. As an example that may also serve as a warning, study the following, which is incorrect, and endeavour to detect the error :

$$\begin{array}{r|l} 4 & \underline{16,\ 18,\ 24,\ 32} \\ 4 & \underline{\ 4,\ 18,\ \ 6,\ \ 8} \\ 6 & \underline{\ 1,\ 18,\ \ 1,\ \ 2} \\ & \ \ 1,\ \ 3,\ \ 1,\ \ 2 \end{array}$$

Ans. $4 \times 4 \times 6 \times 3 \times 2 = 576$ the right answer being half this, viz., 288.

The method is dangerous in careless hands.

PRACTICAL APPLICATIONS OF LCM

As with GCM so with LCM the most obvious use is in the manipulation of fractions, especially in their addition and subtraction. Thus if you desire to add $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, proceed through the LCM of 3, 4, and 5 and add them as $\frac{20}{60}+\frac{15}{60}+\frac{12}{60}$.

A moment's thought will show the close connection between GCM and LCM, for though the process apparently depends upon LCM, yet $\frac{1}{60}$ is actually the GCM of $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$.

Apart from the use of LCM in fractional manipulation, the range of problems involving its use is somewhat greater than that with GCM. Here is a Puzzle Problem :—What is the least possible number of stairs in a flight such that if I go up two at a time or three at a time or four at a time or five at a time, there is always an odd one at the top ?

The answer is clearly—(LCM of 2, 3, 4, 5)+1 i.e., 61 stairs.

The relationship that the product of two numbers is always the same as the product of their LCM and GCM is worthy of note.

TEACHING OF FRACTIONS

The word 'fraction' is from a Latin word meaning "to break". When a bone is fractured, it is broken. Fractions, in arithmetic are used to express parts.

The fractions that children generally use are halves, thirds and fourths. These fractions cover about 90 per cent of common adult usage and about 40 per cent of usage in business. Of course, eighths and twelfths are also commonly used.

How will fractions be taught ?

The child joins the school with a good chance of knowing halves. He may know about the fourth, but seldom about the third. As in life, so in school, let meaning come with use. "Take half of the students to the farm". "Crease your paper so as to show fourths". Initial work should be done with actual objects in terms of circles, squares or oblongs. The additions and subtractions should also be done in meaningful situations. The objective viewpoint is so fundamental that, unless it is secured, the children will be left more or less in confusion, and will not be able to retain the knowledge for long.

The teacher will do well to follow the procedure given below :

Play at fractions until useful relations are seen and known. Have all circles the same size. Mark off into halves, fourths, eighths, sixteenth, thirds, twelfths. These are very useful fractions. A rectangular figure may be used to illustrate the less useful and more difficult

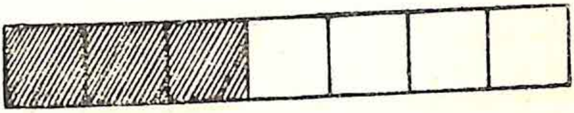


Fig. 11

fractions as shown in the illustration given. The shaded portion shows the fraction $\frac{3}{7}$, i.e., three parts out of seven parts.

ADDITION OF FRACTIONS AND MIXED NUMBERS

The necessity of addition of this type can be explained by the following example :—If your mother buys cloth for the family, she may buy $1\frac{1}{2}$ metres for brother's shirt, $2\frac{1}{2}$ metres for sister's dress, $2\frac{3}{4}$ metres for elder brother's shirt and $2\frac{1}{4}$ metres for herself. She feels the need to add fractions and mixed numbers.

That one half and one half make a whole, can be shown easily by combining the two halves of a circle to make one circle. While dealing with addition of "mixed numbers" containing both whole numbers and fractions, the teacher can similarly depend on whole circles and their fractional parts. The mixed number $2\frac{1}{2}$ can be shown by two whole circles and one half of a circle, while $3\frac{1}{2}$ by three whole circles and one half of a circle. Their addition can be performed by combining the two halves to make one whole and then by combining this whole circle with the other whole circles which are five in number. The answer naturally comes out to be six whole circles or six.

So long as the denominators in the fractions to be added are the same, the addition is not difficult. But when the child has to add fractions with different denominators, the question is different. Suppose he has to add $\frac{1}{2}$ and $\frac{1}{3}$. This addition can be explained, if he divides the whole circle or bar into $2\times3=6$ equal parts. Here 3

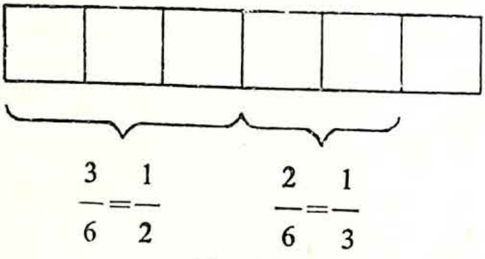


Fig. 12

parts out of 6, will represent $\frac{1}{2}$ and 2 parts out of 6 will represent $\frac{1}{3}$. Combination of the two will give him 5 parts out of 6 parts, that is $\frac{5}{6}$. To convert this objective experience into a quick mental process of solving problems he can say that the whole thing is divided into six equal parts $\left(\frac{\quad}{6}\right)$, for $\frac{1}{2}$, he has to take 3 parts out of 6 $\left(\frac{3}{6}\right)$ and for $\frac{1}{3}$ he has to take 2 parts out of 6 $\left(\frac{2}{6}\right)$. The numerators can more easily be obtained by dividing the denominator 6 by 2 and 3 (former denominators) respectively $\frac{3+2}{6}$.

SUBTRACTION

It can be performed almost on the same lines as addition. When the student has to subtract $\frac{3}{7}$ from $\frac{5}{7}$, the process involves the division of one whole into seven equal parts, then taking 5 parts out of 7 and then diminishing 3 parts from these 5 parts out of 7. The answer left is 2 parts out of 7 or $\frac{2}{7}$. What is to be done when the denominators are different ? Here is a problem : Subtract $\frac{1}{4}$ from $\frac{2}{3}$?

As usual the whole will have to be divided into $4\times3=12$ equal parts.

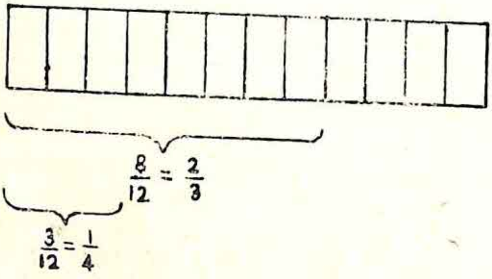


Fig. 13

The teacher can say that the whole has been divided into 12 equal parts $\left(\frac{\qquad}{12}\right)$.

$\frac{2}{3}$ means 8 parts out of $12\left(\frac{8}{12}\right)$ and $\frac{1}{4}$ means 3 parts out of $12\ \left(\frac{3}{12}\right)$. Therefore there are left 5 parts out of $12\ \left(\frac{8-3}{12}\right)$ or $\frac{5}{12}$.

CONVERSION OF A MIXED NUMBER INTO A FRACTION IN A SUBTRACTION PROBLEM

It may be possible to do without conversion where the fraction part of the smaller mixed number is less than the fraction part of the greater mixed number. But when the case is reverse the 'conversion' becomes necessary. Suppose $4\frac{5}{8}$ is to be subtracted from $6\frac{3}{7}$. Here the fraction $\frac{5}{8}$ is greater than the fraction $\frac{3}{7}$, therefore not deductible. So the process of conversion gives the fractions

$$\frac{6\times7+3}{7} \text{ or } \frac{45}{7} \text{ and } \frac{4\times8+5}{8} \text{ or } \frac{37}{8}$$

The denominators are 7 and 8; therefore the whole has to be divided into 56 parts $\left(\frac{\qquad}{56}\right)$. In case of $\frac{45}{7}$ the fraction means 45 parts each equal to $\frac{1}{7}$th or 45×8 parts each equal to $\frac{1}{56}$th i.e. $\left(\frac{45\times8}{56}\right)$.

In case of $\frac{37}{8}$ the fraction means 37 parts each equal to $\frac{1}{8}$th or 37×7 parts each equal to $\frac{1}{56}$th i.e. $\frac{37\times7}{56}$. Therefore, next step

$$\left(\frac{45\times8-37\times7}{56}\right)=\frac{360-259}{56}=\frac{101}{56}=1\frac{45}{56}.$$

MULTIPLICATION

The beginning has to be made with such examples as $\frac{1}{2}$ of 4, $\frac{1}{3}$ of 6, $\frac{1}{4}$ of 8 and so on. These examples can easily be explained by giving them the forms, one half of four, one third of six and one fourth of eight. The fact that denominator of one fraction can be cancelled with the numerator of the other, can be introduced to the students. The next step in this topic will be the solving of problems like $\frac{1}{2}\times\frac{1}{3}$.

This particular problem can be worded as 'one half of one third'. Let the teacher first take one third and then its one half. The total vertical shaded portion is one third of the figure and the lower portion with double shades is 'one half of this one third'. It can be easily seen that the final portion is one part out of six parts, that is $\frac{1}{3}$. Therefore $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$.

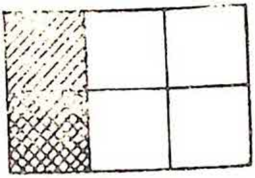


Fig. 14

A more difficult problem such as $\frac{2}{3}\times\frac{4}{5}$ can also be tackled in the same manner. This means 'two thirds of four fifths. The teacher says, "The figure has been vertically divided into five equal parts and we have to take four out of them. Horizontally, the figure has been divided into three equal parts. From the portion $\frac{4}{5}$, we have to take 2 parts out of these three parts. The part of the figure with double shades thus represents $\frac{2}{3}$ of $\frac{4}{5}$. The whole figure has been divided into 15 equal parts and $\frac{2}{3}$ of $\frac{4}{5}$ means 8 parts out of these 15 parts, that is $\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$." With a few more problems of this type, solved in this illustrative manner, it can be easily generalised that while multiplying fractions, the numerator of the product can be obtained by multiplying their numerators and the denominator of the product by multiplying their denominators. Of course the rule of cancelling denominator with the numerator if possible, continues to be applicable.

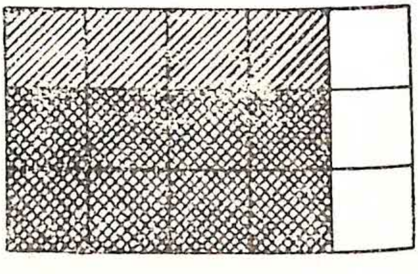


Fig. 15

DIVISION

Here is a simple case of $5\div\frac{1}{2}$. It can be explained by the method of subtraction also. It will be found that 5 contains $\frac{1}{2}$ ten times. But at this stage it will not be difficult for the students to understand that the process of division is secured by inverting the divisor and multiplying. Invert the divisor and multiply, is the universal rule. This rule simplifies the most complex problems to be ultimately attempted in this topic.

DECIMAL FRACTIONS

It was invented in 1584 by Simon Stevin of Belgium. The decimal point was invented by Napier, a Scotchman, in 1617. In the seventeenth century were added decimal fractions.

The ease and speed of modern calculation are due to these inventions : the Hindu notation, decimal fractions and logarithms. Modern invention has added another aid, viz., the calculating machine.

When pupils of primary classes are learning the use of coins and measurement of lengths less than a cm , the idea of decimals can be introduced. But this idea should be exhaustively taught in the middle classes. A clever teacher will try to give the children full understanding as to decimal usage in life generally and in industry particularly.

Newspapers and magazines also often contain some statistical census, governmental and business reports and in them extensive use of decimals is very evident.

A decimal fraction is another way of writing a part or more than a part of a thing. It is merely an extension of the decimal system to fractions. In a decimal fraction you write only the numerator, or number of parts. In decimal fractions very few denominators are used, and they are not written. The denominators are 10, 100, 1000 and others made in this way as powers of ten. These denominators are read tenths, hundredths, thousandths and so on.

Decimals are read and written almost like whole numbers. Here is a plan showing the reading and writing of decimals and whole numbers :

and so on
—ten thousands
—thousands
—hundreds
—tens
—units
—tenths
—hundredths
—thousandths
—ten thousandths
and so on.

The value of each digit increases ten times as it moves one place to the left. In the reverse order, the value decreases ten times as each digit moves to the right. If you continue moving the digits to the right beyond the units', the value of the next one would not be a whole number but $\frac{1}{10}$th of a whole ; similarly the next one would be ten times less, i.e., $\frac{1}{100}$th of a whole.

The topic of decimals as taught in the schools today includes reading and writing, addition, subtraction, multiplication, division reducing common fractions to decimals, reducing decimals to fractions and carrying uneven averages out to two, three, or four places.

Correct reading of decimals should be demanded. Pupils generally read for example, 2.32 as 'two decimal thirty-two'. It is wrong; this should be read as 'two point three two'.

IT IS INTERESTING TO KNOW

1. That one's chief concern is the decimal point; it must never be omitted or neglected.

2. The processes of addition and subtraction in decimals are just like those in the case of whole numbers; but one has to remember the decimal point.

3. That multiplication of decimals goes along exactly the same lines as those of whole numbers, until you come to placing the decimal point in the answer.

ADDITION AND SUBTRACTION

Special attention is to be paid to the proper arrangement of digits. Just as while dealing with whole numbers units are to be placed below units, tens below tens, hundreds below hundreds and so on; similarly tenths are placed below the tenths, hundredths below the hundredths, thousandths below the thousandths, and so on. The carrying and borrowing processes have to be performed in the same manner as in whole numbers. The following examples provide illustrations :—

Addition :—

$$\begin{array}{r} 6785.5867 \\ 457.934 \\ \hline 7243.5207 \end{array}$$

Subtraction :—

$$\begin{array}{r} 9536.4300 \\ 874.8236 \\ \hline 8661.6054 \end{array}$$

MULTIPLICATION AND DIVISION

It will be worthwhile to convert the decimal fractions into ordinary ones in case of some introductory problems for the purpose of understanding. Here are a few illustrations.

1. Multiply 1 and .1.

Conversion gives $.1 \times .1 = \frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$

Now apply decimalisation $.1 \times .1 = .01$.

2. Multiply 62.48 by 53.32.

Conversion gives $62.48 \times 53.32 = \frac{6248}{100} \times \frac{5332}{100}$

$$= \frac{33314336}{10000}$$

Now applying decimalisation $= 3331.4336$

The process can be generalised as :— "In multiplication, the decimal fractions have to be multiplied in the same manner as whole numbers. The position of the decimal point is to be determined by counting total number of decimal places in multiplicand and multiplier.

1. Divide .4 by .1

Conversion gives $.4 \div .1 = \frac{4}{10} \div \frac{1}{10}$

$$\frac{4}{10} \times \frac{10}{1} = 4.$$

2. Divide 11.518 by 4.43

Conversion gives $11.518 \div 4.43 = \frac{11518}{1000} \div \frac{443}{100}$

$$= \frac{11518}{1000} \times \frac{100}{443} = \frac{26}{10}$$

Now applying decimalisation $11.518 \div 4.43 = 2.6$.

The process can be generalised as :— "In division of decimal fractions, the division has to be performed in the same manner as whole numbers. To find the number of decimal places in the quotient, subtract the number of decimal places in the divisor from the number of places in the dividend." But in a sum like 225÷.0025, it is necessary to make sure by adding zeroes to the dividend that there are at least as many decimal places in the dividend as in the divisor.

RECURRING DECIMALS

Recurring decimals are important and interesting. These are needed frequently in the learning of mathematics. Their introduction is based on the conversion of vulgar fractions into decimal ones. They occur when the numerator is not exactly divisible by the denominator in spite of our continuing the division to any number of decimal places.

For example $\frac{1}{3} = 0.3333$.........going on for ever. This recurring decimal is represented by 0.3.

Similarly $\frac{2}{3} = 0.6666$...........going on for ever, is represented by 0·6.

and $\frac{1}{9} = 0.1111$...........going on for ever, is represented by 0.1.

Meaning, understanding, and actual experience in usage are to be emphasised in case of decimal fractions. The need of decimals occurs in ordinary life very frequently. Their occurrence in real problem situations should be exploited fully to give to them added interest and significance. Several units of work involving decimals may possibly be used as illustrations and in certain cases as models. A list of such possible units is given ahead.

1. School bank.
2. Metric system of weights and measures as recently introduced in the country.
3. Using milk.
4. Food values of different articles.
5. Making tooth powder, ink, etc.
6. The school fund.
7. The composition of air.

METRIC MEASURES

How did this metric system originate ? To understand how a tried system of weights and measures could be replaced overnight in the country, it is necessary to understand the background.

The metric scheme is a product of the French Revolution. The leaders of the revolution were followers of Rousseau. They constructed a new calendar, in which Sept, 23, 1792, the date of the proclamation of the French Republic, was fixed as the first day of year 1. The length of the week was changed to 10 days. It was decided that the meter, the name for the new unit of length would be the basis of the entire system of weights and measures and that its value should be fixed as one ten-millionth part of the distance from the equator to the pole on the meridian passing through Paris. This was to be the "unit from nature". In their youthful haste, the National Assembly of France adopted a measure for meter in 1793, changed it in 1798, but even then it was later found to be inexact.

Finally, the values of the units were agreed upon, and the system was introduced by law. The change did not happen overnight. Advanced countries had been contemplating this change for the past five centuries. The necessity of a uniform system of weights and measures had been felt all over advanced countries. Under these circumstances, the leaders of the French Revolution did not hesitate to abolish the old "man-made" scheme of weights and measures and substitute a new one based upon a "natural unit".

While introducing the new system, the National Assembly of France banned the other systems and imposed penalties. The result was, of course, a complete failure. The practical-minded Napoleon came to the rescue in 1812. He legalised the use of the "useful units" in trade. He retained the teaching of the metric system in the schools, and the use of the metric system in official reports. In 1840 again, after 45 years of official use and teaching in the schools, the old measures were strictly forbidden in the trade and a scheme of severe penalties was adopted. Even then the non-metric measures survived for a pretty long time. It is only recently that a more convincing and congenial atmosphere has been created for the metric system.

Briefly speaking, these changes may be listed as follows :—

Metre, instead of yard.
Millimetre, instead of inch.
Litre, instead of quart.
Kilogram, instead of a seer.
Kilometre, instead of mile.

The claims of the metric system are :—

1. The basis, a "unit in nature."
2. Close inter-relationship of the different tables (length, weight, capacity).
3. Complete decimalisation, which is the very corner-stone of the metric system.

These claims read well, and also make a strong appeal to the academic mind, but the new system is not considered indispensable and essential by many. To some, the customary units appear more convenient and better suited to trade conditions than the metric system is. They have remained in use in many countries because of their inherent merit and in spite of legislation against them. Many manufacturing and trade concerns have to this day remained largely non-metric. It can reasonably be said that official patronage and propaganda in favour of the metric system should cease and that it should compete unaided with the customary units.

One system of measures for the entire world is often advocated, and theoretically it appears to have advantages. But at the same time one must not be in a great hurry about it. One should not forget that measures are a social heritage, and, in certain cases, are as difficult to change as a language. The attitude to metric measures should be rational—let the system be adopted on merit alone.

DECIMAL SYSTEM IN INDIA

Government of India, in its move to replace the non-metric system by the decimal system, introduced decimal coinage in India with effect from 1 April 1957. The rupee remains the same in value and nomenclature. It has been divided into 100 equal parts called 'Naya Paisa' (later called Paisa) instead of its previous division into 192 pies. By introducing decimal coinage, India has carried out a reform in currency which has already been adopted by most of the coin-issuing countries of the world.

Although the new coinage will be on a decimal basis, there is no orthodox rigidity of the decimal principle in it. According to rigid definition, the various denominations of coins have to be arranged in multiples or sub-multiples of ten with reference to a standard unit. Thus the higher coins will be 10, 100, 1000 etc., and the lower units will be .1, .01, etc. But in the Indian coinage system, starting from 'Paisa' as the unit, there have been retained also other multiples of 2, 5, 25 and 50. In this way are being retained some of the familiar and popular coins. This mixed form of coinage is a necessity in a country like India with appalling illiteracy among the masses that consequently need a long transitional period to get used to even a minor change in the system of measures.

In order to make the decimal coinage effective, it is being linked with the introduction of the metric system of other measures. Decimalisation has been acclaimed all over the world as the simplest form of weights and measures. In the modern complicated system of trade and commerce, easy and quick conversions are most essential. The decimal system, that works on multiples of ten, makes the task of conversion easier.

In almost all towns the changeover to metric weights is being seriously implemented. The traditional weights will persist in villages for a longer time. Retail business still uses old weights in terms of new weights. The use of old weights has, of course, been prohibited by law. These weights and their use are becoming extinct with the passage of time. The new measurements of length are the latest introduction which has not become familiar and popular as yet.

A complete changeover to metric systems is long overdue in school and college syllabi and text-books. Old measures should no longer exist in them. Thus the future generation can bring about a full-fledged changeover in all respects.

SQUARE ROOT

What do we mean by "Square Root ?" It is the reverse of 'square'. It is the number that when multiplied by itself gives the 'given number'.

The necessity for finding square root continually occurs in all forms of practical mathematics. Simple examples will have already been set in connection with factor exercises. Thus the student may find by simple factors the square root of perfect squares such as

121, 144, 196, 256, 441, etc., though he could hardly be expected to find by factors the square root of such a number as 5329 or (73×73).

Again, by comparison with known squares the student will be able to approximate roughly to the square root required. Thus he can see that $\sqrt{20}$ lies between 4 and 5 since 20 is between 16 and 25. He may also reasonably assume that the root would be very nearly 4.5 (since 20 is approximately half way between 16 and 25), and he is not far wrong. Such estimation and rough approximation should be encouraged as much as possible. Similarly he should be asked to note that $\sqrt{260}$ is not much above 16; and if he tries $16\frac{1}{4}$, he will find that he is nearing the answer $(16\frac{1}{4}\times16\frac{1}{4}=256+8+\frac{1}{16})$. Let him try 16^1. $(16\frac{1}{8}\times16\frac{1}{8}=256+4+\frac{1}{64})$, he will find that he has almost got the correct answer.

But finding of square root to any desired degree of accuracy still remains to be done. The traditional and popular process is easily explained on the basis of the identity $(a+b)^2=a^2+2ab+b^2$. Suppose the square root of 4356 is to be found out. The square 4356 is represented by fig. A. The next step is to take out the largest possible square (say 60^2), i.e. 3600 (fig. B). This leaves out figure C containing 756. This is further converted into a rectangle D. Clearly 756 has to be split into two factors $120+x$ and x, and by trial it is quite possible to find $x=6$. Therefore 4356 is the square of 60+6 or 66.

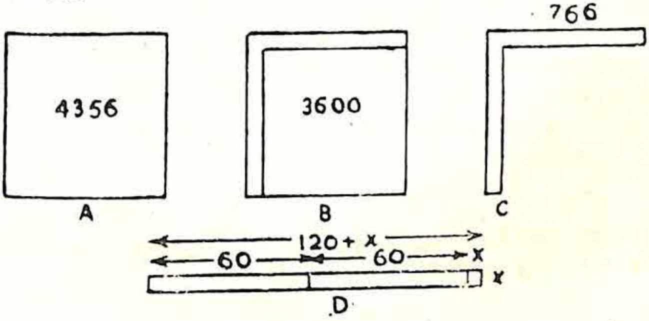


Fig. 16

The traditional method as given below can be explained on the basis of the above graphic representation :—

$$\begin{array}{r|l} & \underline{\;\;66} \\ 6 & 4356 \\ & \underline{36\;\;\;\;} \\ 126 & \;\;756 \\ & \underline{\;\;756} \\ & \;\;\times \end{array}$$

All cases, integral or fractional, can be explained by similar diagrams. This graphical explanation of the general method is as old as Ptolemy, but has been re-discovered and elaborated with considerable ingenuity by later writers, notably Dr T.P. Nunn.

This method applies equally to decimals as to whole numbers.

GEOMETRICAL METHODS

A simple geometrical method combines in it the Theorem of Pythagoras, and that of the identity $x^2-y^2=(x+y)\,(x-y)$. Any given number may be expressed as the sum or difference of the squares of two numbers. The natural numbers may be written in one row and their squares in the second

1	2	3	4	5	6	7	8	9	10
1	4	9	16	25	36	49	64	81	100

The relevant numbers can be matched, just as :—

$$\sqrt{7}=\sqrt{16-9}=\sqrt{(4)^2-(3)^2}$$
$$\sqrt{11}=\sqrt{36-25}=\sqrt{(6)^2-(5)^2}$$
$$\sqrt{10}=\sqrt{9+1}=\sqrt{(3)^2+(1)^3}$$

and so on.

THE "MEAN PROPORTION" METHOD

This graphical method of finding square roots is equally fascinating to young learners. The principle is best introduced with the usual semicircular diagram where it is known that the rectangle $a\times c$ is equal to the square on b, as shown in the opposite diagram :-

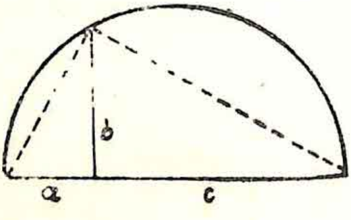


Fig. 17

This truth may be investigated by actual measurement or may be proved in the orthodox way.

PRACTICE

The method of *practice* came into use in the sixteenth century, when it was introduced by the merchants and negotiants of Italy for expediting business calculations. Although the early writers laid stress on the utility of the method in business calculations, it is clear that they also realised the general utility of the method as a method of multiplication in general. Practice is best viewed not as a new type of problem but as an alternative method of obtaining results of a multiplication. It follows, that, as a method it may be applied to all multiplications and is not to be confined solely to costs.

The process consists of selecting such a system of units for the multiplier as will enable the learner to convert it into a fraction or series of fractions, and thus to attain, the result by the simple process of short division. Division is generally easier than multiplication, and, because the operations of division proceed from left to right in our notation, the process removes any difficulty of determining the position of the decimal point; it also facilitates approximation.

It is customary to begin the teaching of the Practice method with much drill in 'parts', and 'parts of parts'. Thus the learner needs such constant practice as is given below :—

What part of Rs 16 is Rs 4 ?
What part of Rs 9 is Rs 3 ?
What part of Rs 3 is Paise 50 ?
What part of Re 1 is Paise 20 ?
What part of Paise 10 is Paise 2 ?

The beginner needs considerable practice in splitting any quantity into parts of a larger quantity.

While setting examples in Practice one must bear in mind the note of warning already Sounded—i.e. examples involving large numbers and very complex quantities occur but seldom in actual life. It is wise to confine problems to simpler quantities and real cases.

The method of practice is applicable to problems on taxes, duties, dividends, bankruptcy, simple interest, compound interest, decimals percentages and costs.

Example—A bankrupt owes me Rs 1265.52, but pays only Paise 73 in the rupee. What do I receive ?

	1265.52
50 Paise = $\frac{1}{2}$ of a Re	632.76
20 Paise = $\frac{1}{5}$ of a Re	253.10
2 Paise = $\frac{1}{10}$ of 20 Paise	25.31
1 Paisa = $\frac{1}{2}$ of 2 Paise	12.66
	Rs 923.83

THE UNITARY METHOD

What is meant by Unitary Method? The usual dependence on some problem situation is the best way out. "A dozen pencils cost Re 1.56 Paise. Find the price to be paid for 2 pencils?" The solution proceeds as :

Cost of 12 pencils = Re 1.56
Cost of 1 pencil = 156 ÷ 12 = 13 Paise
Cost of 2 pencils = 13 × 2 = 26 Paise.

The unitary method is the process of finding the cost of a required number of articles, when the cost of a number of articles has been given. That is, if you know the value of a number of units you can find out the value of one unit by division and by multiplication the value of any number of units. The cost of most of the things you buy, is reckoned from the cost of the unit of that thing. Hence the unitary method is mostly a combination of division and multiplication. Every problem consists of two parts, the given and the required, or the statement and the question.

Example :—If a car travels 660 Kilometres in 12 hours, how far will it travel in 14 hours ?

The first thing is to ask, "What is the ultimate question?" It is "How many Kilometres?" Then consider the wording of the last line and arrange it so that the number of Kilometres comes at the end. It will be "In 14 hours the car travels—Kilometres". Now word the given statement accordingly, i.e. "In 12 hours the car travels 660 Kilometres. "In 1 hour the car travels $\frac{660}{12}$ Kilometres.

∴ in 14 hours the car travels $\frac{660}{14} \times 14$ Kilometres=770 Kilometres.

This method should be used generally for 2 or 3 years; but later on it may be shortened thus :

In 12 hours the car travels 660 Kilometres

In 14 ,, ,, ,, ,, 660×—Kilometres

The teacher will utilise the previous experiences from which the students know that 660 Kilometres has to be multiplied by one of the numbers 12 and 14 and divided by the other. Has the number of Kilometres to be made larger or smaller ? This will bring the answer that it has to be made larger. How can this number be made larger ? The natural answer will be—"by multiplying it with the larger number and dividing by the smaller." Thus the final step can be put down as $660 \times \frac{14}{12}$.

The unitary method forms the background or foundation of most of the topics to come. It should, therefore, be understood and practised thoroughly.

RATIO

What is ratio ? It is the number which gives the relation of a certain quantity to another quantity; for instance, the ratio of a 5 Kgm. weight to one Kgm. is 5, because the former is five times as heavy as the latter. Similarly the ratio of 20 to 5 is 4. The concept of ratio is related to that of fraction. Applying the Fraction

Method, the above examples can be rewritten as $\frac{5\text{ Kgm}}{1\text{ Kgm}}=5$ $\frac{20}{5}=4.$

The writing of ratio is done in particular ways. The ratio of 20 to 5 can be written as 20 : 5 or $\frac{20}{5}$ or 20÷5. Its simplified form will therefore be 4 : 1.

An interesting experiment with 12 beads or balls can be useful in this respect.

(*i*) Take one bead and place it away from the remaining 11. Ask the student the ratio of these two quantities. It is 11 : 1.

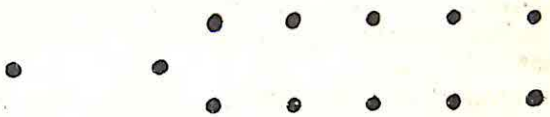


Fig. 18

(*ii*) Take two beads and place them away from the remaining 10. Ask the students the ratio of these two groups of beads. It is 10 : 2. But 2 beads are present 5 times in the group of ten. The ratio can also be written as 5 : 1.

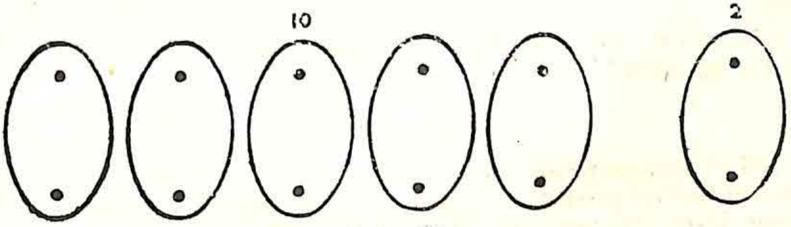


Fig. 19.

(*iii*) Take 3 beads and place them away from the remaining 9. Ask the students the ratio of these two groups of beads. It is 9 : 3. But 3 beads are present 3 times in the group of 9. The ratio can also be written as 3 : 1.

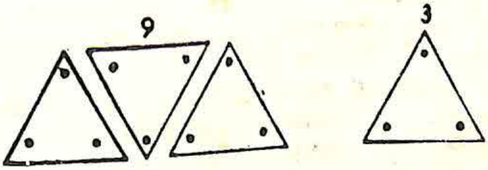


Fig. 20

(*iv*) Take 4 beads and place them away from the remaining 8. Ask the students the ratio of the two groups of beads. It is 8 : 4. But 4 beads are present two times in the group of 8. Therefore the ratio can also be written as 2 : 1.

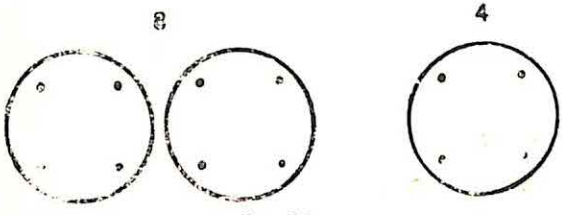


Fig. 21

A few such problems can enable the students to understand the significance and process of ratio. It can be easily made clear to them that two quantities are compared by dividing the first by the second; hence in a ratio the quantities compared should be of the same kind, e.g., three pencils can be compared with one dozen pencils and not with 6 books.

Simplification of a ratio like 30 : 20 can be easily explained by converting it into its fractional form as $\frac{30}{20}$, and then cancelling the common factor of the numerator and denominator. With a little practice in this manner, an average student is sure to understand that he can straightaway cancel the common factor in the two quantities of a ratio. The converse can also be brought home to the students. They can be told that the terms of a ratio may also be multiplied each by the same number without altering its value.

The writing and simplification of the ratio of more than two terms is governed by the same rules.

PROPORTION

What is Proportion ? When four terms are so related that the ratio of the first to the second is the same as the ratio of the third to the fourth, they are said to be in proportion.

Knowledge of 'ratio' and a clear idea of fractional multiplication and division are essential prerequisites of proportion.

The idea of proportion may be developed by giving examples such as 1 : 2 is the same as 2 : 4 or 1 : 3 is the same as 2 : 6 or 2 : 5 is equal to 4 : 10. Their fractional forms can make the idea still easier. A statement in proportion is a statement of the equality of two ratios. The proportion is written in a particular manner as 1 : 2 : : 2 : 4, or 2 : 5 : : 4 : 10.

The first and the fourth terms are called the extremes ; the second and the third terms, the means or middle terms. The fourth term is also known as the fourth proportional.

The teacher will give them the opportunity of establishing and verifying the statement that the product of the extremes is equal to the product of the means.

Their attention will be drawn to the example already given, viz., 1 : 2 : : 2 : 4. In this case the second term 2 has occurred twice. This repeated term is called "mean proportional". The last quantity in this case is called "third proportional" and not the fourth, because there are in all three terms in this proportion. When the ratio of the first to the second is the same as the ratio of the second to the third, the three quantities are said to be in continued proportion. For example, 12 : 6 : : 6 : 3. Therefore 12, 6, 3 are in continued proportion. Here 6 is the mean proportional between 12 and 3. The students will be made to observe that the mean proportional between two numbers is equal to the square root of their product.

At this stage, it will be quite appropriate to give them the idea of direct and inverse proportions. When two quantities given in a proportion, increase and decrease together, it is the case of direct proportion. For example,

Time	Distance
1 hour	4 Kilometres
2 hours	8 Kilometres
3 hours	12 Kilometres

Similarly, the following can be quoted as example of direct proportion :—numbers of articles and their costs, circumferences and radii; wages and hours of work ; mileage and fares.

INVERSE PROPORTION

Here, if one quantity increases, the other decreases. For example, the area of a rectangle is 84 sq. ft., the length and breadth or sides are in inverse proportion as :

One side	Other side
42	2
28	3
21	4
14	6
12	7

There are other examples of the type : "supposing a particular quantity of food is sufficient for 200 men for 20 days, the proportionate table will be :

Men	Days
200	20
100	40
50	80
25	160
400	10
800	5

Proportions can likewise be simple or compound.

A CASE OF SIMPLE PROPORTION

At 60 Kilometres an hour a train took $3\frac{3}{4}$ hours. How long would it have taken at 80 Kilometres an hour ?

A CASE OF COMPOUND PROPORTION

45 men could build a wall in 30 days, working 12 hours a day. How many days of 8 hours each would 60 men take ?

At least four methods are available for the effective teaching of proportion. Taught any of these four methods children soon attain considerable skill and accuracy with straightforward examples—

(1) *The Traditional Method* depending upon extremes and means: "If 5 books cost Rs 7.85, what would 11 cost ?" The putting of dots can straightaway lead to its solution—

$$5 : 11 : : 7.85 : x$$

(2) *The Method of Unity*. The advocates of this method claim that it is reasonable, based on common sense, easily taught and capable of simple explanation. But it is hardly a suitable method, since to proceed always through the "unit" and to write out the three lines of statement is often cumbersome and even unnecessary.

(3) *The Fractional Method* : Here the unit line of the previous method is discarded and the final statement of the answer in correct form is obtained in one step, calculation only being employed to simplify the answer. For example—If 12 metres of carpet cost Rs 56.40, what would 18 metres cost ?

$$\text{Ans } \frac{18}{12} \text{ ro } \frac{3}{2} \text{ of Rs } 56.40$$

The students must be taught to reason along the same simple lines, viz.,

(a) Is my answer a number or money or a quantity, etc. ?
(b) Is it more or less ?
(c) In what proportion is it more or less ?

Thus it would appear to combine the Method of Unity and the Fractional Method, and not to require the students to adhere slavishly or mechanically to either of the two methods. The merits of the problem should determine the method. The unit line should never be used when it can obviously be avoided. To attempt to teach the one or the other as the rule to be always followed, is mechanical and cramps thinking and reasoning.

(4) *The Equational Method.* This method of setting down the results of reasoning in comparative arithmetic is simple and useful.

Take the example : "A map scale was 12 Km. to the inch. What length of line represented a distance of 42 Km. ?" It might be set out as $\frac{x}{1} = \frac{42}{12}$.

It is a good method. It is readily taught and mathematically sound. The process of solution is sensible, and, at the same time, speedy and accurate.

To sum up, the teacher has to avoid any method that is ill-understood by the average child even though that may be his choicest method. He should rather give a free scope to every child's mathematical common sense. If he can "jump" to the answer, let him, otherwise let him proceed in his own way and at his own pace. This attitude will be more profitable in the end. People who can think accurately in comparisons are of more value to the world at large than those who can patiently and obediently work in a mechanical manner.

HOW TO SOLVE COMPOUND PROPORTION

The attention of the students will be drawn to the problem of compound proportion already given—45 men could build a wall in 30 days, working 12 hrs. a day. How many days of 8 hrs. each would 60 men take ?

If children can readily apply the fractional method, this presents no new difficulty at all but is simply a combination of a series of simple proportions, sometimes direct, sometimes inverse.

Thus it can be solved as

45 men working 12 hours a day take 30 days
60 ,, ,, 8 ,, ,, ?

It can be further argued thus

Original number of days=30 days.

(a) More men $\therefore$ less days, i.e., $\frac{45}{60}$ of 30 days

(b) Less hours per day, $\therefore$ more days i.e., $\frac{12}{8}$ of $\frac{45}{60}$ of 30 days.

The knowledge of proportion is very much applicable to similar triangles, areas of similar figures and the value of similar solids.

PARTNERSHIP

Partnership is a process by which persons join their efforts and funds to run a joint business. Partners hold different shares in the capital of investment.

Actual business situations can provide scores of examples to illustrate partnership. Suppose a person has a plan to start a flour mill, but has only Rs 1200. The total investment required is about Rs 5000. He contacts one of his relatives to become his partner. The relative agrees, but can invest at the most Rs 2000. Consequently he has to request one of his friends to make the remaining investment. At the end of the year, the mill brings a net profit of Rs 1500, which has to be shared by the shareholders.

The knowledge of ratio and its simplification is the prerequisite of this topic, because ultimately the profit has to be shared among the shareholders in the ratio of their investments.

If the time of investment in the case of each partner is the same, whether the capitals invested are equal or not, the partnership is known as 'simple partnership' and the profit or loss is distributed in the ratio of their capitals.

But if the time for which capital is used is not the same in all cases, whether capitals contributed by the partners are equal or unequal, the partnership is known as 'compound partnership'; in such a case both the capital and the time are to be taken into consideration.

The example given before is that of simple partnership.

The solving of problems of compound partnership involves the multiplication of the sum invested with the number of months or years for the purposes of creating a reliable comparison between different investments and establishing a reasonable ratio between them. Sometimes compound partnership contains further intricacies, as in the following cases :— (i) A person invests a particular sum in the beginning, but withdraws a part of it after a few months or adds something more to his investment during the course of the year. (ii) A shareholder even joins a business after it has already continued for sometime on the basis of the capital contributed by others. (iii) One of the partners is also the employee of the concern and the others are sleeping partners. A little of caution and alertness on the part of the learner can easily enable him to overcome such difficulties. He has only to follow the statement of the problem as faithfully as possible.

TEACHING OF PERCENTAGE

Percentage is a basic idea in business, and is important enough to justify acquainting children with it fairly early. It means out of hundred, or per hundred. It represents fractions and ratios where the denominator is always a hundred. Daily there are references involving percentage, e.g., per cent of increase, per cent of discount, rate per cent on loans and deposits, percentages of various mixtures, pass percentage of students etc. One can often find notices, regarding some per cent reduction in prices, hanging outside various shops. It is a common phenomenon in the sale of articles by *khadi bhandars* and stores selling handloom cloth. The topic can easily be introduced to the class by referring to its use in daily life.

'The hundred square' is a very useful illustration of percentage. The square consists of 100 small squares. One out of these hundred squares represents 1%. A few more percentages such as 2%, 3%, 5%, 10% and 14% are shown in the diagram.

The term 'per cent' is used in profit and loss, interest, discount and many other topics. Its value can further be realised by referring to its application in sciences, hygiene, geography, domestic economy etc.

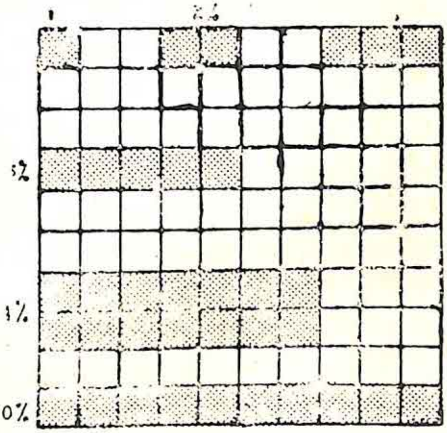


Fig. 22

REDUCING A FRACTION TO A PERCENTAGE

It is a very important aspect of the topic of fraction. It should be emphasised that 100 is used as the standard denominator in order to compare the values of two or more fractions. Here is a problem "Suppose a person purchases a buffalo for Rs 300=00 and sells it for Rs 360=00. Another person purchases one for Rs 250=00 and sells that for Rs 290=00. Which is the better bargain ?" Naturally the student has to find out the gain per hundred in both cases, which will enable him to know the better bargain out of the two.

The gain for Rs 300=00 is Rs 60=00

The gain for Rs 1=00 is $\frac{60}{300}$

The gain for Rs 100=00 is $\frac{60}{300}\times 100$ or 20%.

Similarly in the second bargain, gain for Rs 100 is

$$\frac{40}{250} \times 100 \text{ or } 16\%.$$

So the first bargain is the better one.

Percentage is naturally an extension of decimal fractions. The operations involved are not new operations but merely new applications of familiar practices. Whereas in the general treatment of decimals there are different units to be considered, in the study of percentage there is concentration upon only one, viz., hundredths.

There are three cases of percentage :—

(*i*) To find a fractional part of a given number.
(*ii*) To find the ratio between two numbers.
(*iii*) To find a number when a fractional part of it is given that is (*i*) What is 5% of 500 ? (*ii*) What per cent of 400 is 50 ? and (*iii*) Three per cent of a number is 12, find the number ?

To make the students realise the value of percentage, problems and data should be as far as possible taken from geography, science, hygiene etc.

PROFIT AND LOSS

What is meant by profit and loss ? It may be safely assumed that the student of today must have learnt the significance of profit and loss from his environments. Every bargain usually involves either of the two.

The knowledge of percentage is the prerequisite of the study of this topic.

The topic may be introduced effectively with the help of a few oral problems.

GIVE AN EXAMPLE LIKE THIS

A milk-seller buys a buffalo for Rs 800=00 After a year when it gets dry he sells it for Rs 500. A student with common sense can conclude in no time that the milk-seller suffers a loss of Rs 300. Similarly from the bargain that a businessman buys a buffalo for Rs 600 and sells it for Rs 800, the student can easily conclude that the businessman gets a profit or gain of Rs 200.

It will not be difficult to make the pupils realise that—

Gain=Selling price-Cost price; and

Loss=Cost price-Selling price.

While calculating Gain or Loss %, it should be made clear to the students that profit or loss is always reckoned on cost price Cost price is the investment of the person who gains or loses. Nothing but investment can be the basis for calculating profit or loss. Finding out profit or loss % is a simple case of application of the unitary method. Total gain or loss is on total investment. The question is : what would the person gain or lose by investing Rs 100 only ?

AVERAGE

Average provides an approximate representative of a group. With its help it can be said 'This saree is cheap' or 'This is a small car', or 'He is a fast runner', or 'Mr Kanwal is a rich man'. In all these cases you are comparing the quantities and qualities with their respective averages.

In a basic school, it is a common experience that speed in spinning of a body varies from day-to-day on account of many factors. Suppose he works for half an hour a day, and spins 20 rounds on Monday, 28 rounds on Tuesday, 30 rounds on Wednesday, 29 on Thursday, 32 on Friday, and 31 on Saturday. If he is required to tell his speed in spinning, he naturally needs to calculate the average of the rounds spun on different days of the week.

A shopkeeper's income also provides a good problem for average. His monthly income is never constant, but he knows well his approximate monthly income. Similarly a farmer makes an estimate of his annual income by applying the principle of average.

The average attendance of a class can be an interesting activity for the learners. If farming is being reasonably pursued as a craft, the average yield per day of different vegetables or average yield per acre of different crops can be interesting instances.

There is nothing very technical about the process of average. It involves only the addition of different terms and then dividing this sum by the number of terms. All this is quite evident and a matter of common sense.

INTEREST, SIMPLE AND COMPOUND

SIMPLE INTEREST

What do we mean by Simple Interest ? Its significance can be easily shown by a chart where a person, in the first picture borrows a certain amount from a money-lender, and, in the second picture has to pay something in addition at the time of clearing the accounts. Some people have surplus money, while others are short of it. The latter have to borrow money and pay some extra money for the borrowed money. This extra payment for the use of money or loan is called interest.

If a boy understands that 5% of a thing is $\frac{5}{100}$ of it (and he is expected to have learnt it from other work on percentages and knows the unitary method, he should have no difficulty with simple interest. Its knowledge can be easily introduced and effectively developed from the previous knowledge of percentage and the unitary method. Some general information about interest can be collected by the children themselves from the banks, post offices, small saving authorities and business agencies.

Initially the teacher will help the students to solve the problems by S.I. by the unitary method; such as

Ex. Find the simple interest on Rs 500 for 3 years at 5% per annum ? (N.B. Don't forget per annum).

Interest on Rs 100 for one year=Rs 5

$$\text{,,} \quad \text{,,} \quad \text{Re } 1 \quad \text{,,} \quad \text{,,} \quad =\text{Rs } \frac{5}{100}$$

$$\text{,,} \quad \text{,,} \quad \text{Rs } 500 \quad \text{,,} \quad \text{,,} \quad = \frac{5}{100} \times 500$$

$$\text{,,} \quad \text{,, Rs 500 for three years} = \frac{5 \times 500 \times 3}{100}$$

So, in generalised form, the formula can be put as :

$$\text{S.I.} = \frac{\text{Principal} \times \text{Rate} \times \text{Time}}{100} \quad \text{or} \quad \frac{\text{P} \times \text{R} \times \text{T}}{100}$$

After sometime, when the students get acquainted with the solving of equations, the formula can be given a new and more useful form, i.e.,

$$100 \times \text{S.I.} = \text{P} \times \text{R} \times \text{T}.$$

This formula can cover all other formulae on simple interest which have to be learnt and used later for solving the inverse cases of finding principal, rate and time.

COMPOUND INTEREST

What is meant by Compound Interest? There should be no difficulty about explaining its significance. Money is said to be borrowed on compound interest when at the end of a year, the due interest is not paid by the borrower, but gets added to the sum borrowed and the amount thus obtained becomes principal for the next year. The same process, i.e., addition of the yearly interest to the previous amount, is repeated until the amount for the last year has been obtained. The difference between the original sum borrowed and the final amount is the required compound interest.

The mention of its evil in our villages will also not be out of place. The teacher will enable the learner to realise how amount accumulates at a surprising and unimagined rate, when interest on interest is charged year after year.

It is obvious that the knowledge of compound interest will have to be developed on the basis of that of simple interest. The teacher will draw the attention of the learners to the formula for calculating amount, which they must have learnt during work on simple interest.

$$\text{Amount} = P\left(I + \frac{R.T.}{100}\right)$$

This formula will enable the learner to find out the value of amount after, one, two, three, four or any number of years. It will be possible for him to find that

$$\text{Amount after one year or } A_1 = P\left(1 + \frac{R}{100}\right)$$

$$\text{Amount after 2 years or } A_2 = P\left(1 + \frac{R}{100}\right)^2$$

$$\text{Amount after 3 years or } A_3 = P\left(1 + \frac{R}{100}\right)^3$$

... ...

$$\text{Amount after } n \text{ years or } A_n = P\left(1 + \frac{R}{100}\right)^n$$

After this stage the finding of the compound interest requires only the subtraction of original sum or principal from the final amount, i.e.,

$$\text{Compound Interest} = P\left(1 + \frac{R}{100}\right)^n - P$$

or $$\text{C.I.}=P\left[\left(1+\frac{R}{100}\right)^{n}-1\right]$$

The remaining work is only the efficient application of this formula.

DISCOUNT

WHAT IS DISCOUNT ?

It is another name for concession, which a customer gets while buying something from a shopkeeper. Newspapers and magazines often carry advertisements of discounts. Such advertisements may be collected by the pupils for reference at the proper time. They may be directed to find the reasons for the concessions or discounts.

The discussion should, of course, lead the students finally to the topic i.e., trade discount.

They can find for themselves that certain firms sell articles on credit or on instalments-basis. They charge something extra from the customers for this facility. An agreement is signed between the customer and the businessman. It is agreed that for an article of the value of Rs 200 the customer will pay Rs 220 after a year. But if by chance the customer received money after a few days from some unexpected source, he may then like to make the payment before the end of the year. According to the agreement he has to pay Rs 200 after a year. But if he can pay now, what concession should he get? Naturally the amount written in the agreement is in excess of the actual or present value of the article. The excess, which will now be left as discount is 10% of the present worth (P.W.) The students can be led to arrive at the generalisations :—

Amount = P.W + Discount.
Discount = Interest on P.W.

In the above problem Rs 20 is the discount to be deducted, Rs 200/- the present worth of the article, Rs 220 the amount written in the agreement as value of the article after adding to P.W., the interest on P.W.

TIME AND WORK

It is an important and useful topic. It deals with the inter-connection between the amount of time devoted and the amount of work accomplished, or the number of workers and the work done.

In every sphere of life, labour is engaged to accomplish a piece of work in time. The example of a contractor best illustrates the significance of this topic. Suppose a contractor engages 200 labourers to complete the construction of a five mile road in 3 months. At the expiry of one month, he finds that only one fourth of the work has been accomplished. By how much should he increase his labour force to finish the work in time ?

The work on 'Time and Work' is more or less a repetition of the processes of 'Unitary Method' and 'Ratio Proportion' though, in a slightly new form. The teacher is not required to introduce any new method for its teaching. An active knowledge of the above mentioned topic will ensure efficiency in this new topic.

While dealing with compound cases of Time and Work, the teacher should refer to the fractional method employed for compound proportions.

AREA

First of all, what do you mean by area ? Perhaps, no definition will make the idea clearer; statements, such as "The area of a rectangle is the space enclosed by its sides", are quite unintelligible. Also, to start with a definition is very defective teaching. But you cannot avoid the definitions for an indefinite period. Let there be a comparison between 'measuring a length' and 'measuring a surface.' Can they measure the surface or the area of the blackboard ? The students have by now come to know that there should be a unit for every type of measurement. Weights, money, time and lengths have all got their respective units. This will lead them to realise that, for measuring a surface, they need a unit. Only then would they be able to find out how many times that units contained in the surface or area of the blackboard.

The students may ask, "Why do we find area ?" The teacher should be able to answer this question easily. Let him refer to a number of situations such as measuring area of a room, surface of a table, a ground or a farm.

At this stage, the measures, 'square centimetre', 'square inch', etc., should be presented before them as realities. The cardboard pieces of these unit areas should be handled and examined by them. For square yard, a cardboard illustration may become bulky, but by drawing a square yard on the blackboard, the pupils can be made to discover that 9 sq ft.=1 sq. yard. Similarly they can be made to realise that 144 sq. inches=1 sq. ft.

Coming back to the problem of finding the surface of the blackboard, the students can be asked to measure the units of area contained in it with the help of the unit of 'square foot'. They will discover that the unit of 'square foot' is contained in it 15 times. Naturally, the area is 15 sq. ft.

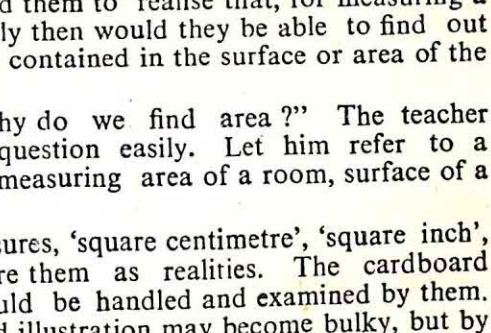


Fig. 23

Similarly a few more problems can be tackled in this manner. With the help of a sufficient number of concrete cases, the students can be led to the conclusion, Area = Length × Breadth (of course, in case of rectangular figures). Area of a square can be easily based upon this formula.

There are also certain reverse problems that is, problems in which area is given, either length or breadth is given, and the unknown dimension is to be found out. The unitary method can be applied to such problems. There is another type of problems wherein the area of the rectangle and the ratio of length and breadth are given and the length and the breadth are to be found out. For example—"The area of a rectangle is 2160 sq. ft. Its length and breadth are in the ratio of 5 : 3. Find the length and breadth." The solving of this problem by the method of algebra is very simple. If the length is $5x$, breadth is $3x$. Therefore area $5x \times 3x$ or $15x^2 = 2160$ or $x^2 = 144$ sq. ft., and so on. Otherwise also, the students can be made to understand that there is some common factor. Length is 5 times that unit and breadth 3 times that unit. Divide the length into 5 equal parts, the breadth into three equal parts, and then the figure into fifteen parts as shown in the diagram. Thus the whole rectangle gets divided into 15 square pieces. The area of each square piece $= \frac{2160}{15} = 144$ sq. ft.

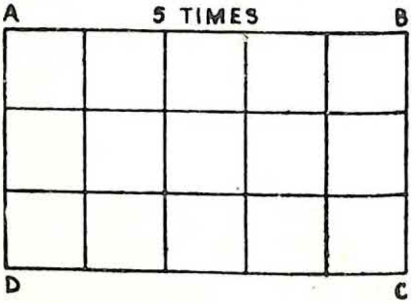


Fig. 24.

The side of each square piece $= \sqrt{144} = 12$ ft.

Length contains 5 such sides and breadth contains 3.

$\because$ Length $= 12 \times 5 = 60$ ft.
Breadth $= 12 \times 3 = 36$ ft.

AREA OF RIGHT-ANGLED TRIANGLE

Example : Of a rt. $\angle d \triangle$ the base $= 6''$ and perpendicular $= 3''$ find the area ?

It requires the completion of the rectangle ABCD as shown in the diagram. By folding this rectangular piece along AC it can be straightaway show that the rt. $\angle d$ $\triangle$ABC is half of the rect. ABCD.

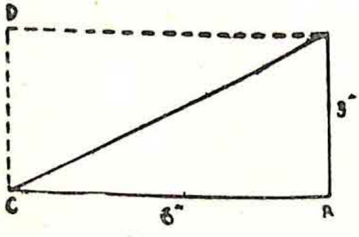


Fig. 25.

$\therefore$ Area of $\triangle ABC = \frac{1}{2} \times L \times B = \frac{1}{2} \times 6 \times 3$.

The dimensions 6 and 3 are nothing but the base and perpendicular respectively of the $\triangle ABC$.

$\therefore$ area of $\triangle ABC = \frac{1}{2} \times$ Base $\times$ Altitude.

AREA OF TRIANGLES

Similarly in case of other triangles the triangle can be converted into a rectangle which may be double in area but with the same base and altitude, or equal in area but with the same base and half the altitude.

AREA OF PARALLELOGRAM

[A parallelogram is only a slightly twisted rectangle. By cutting the portion BCC' and placing it as ADD', it is ascertained that the ∥ gm ABCD has been converted into the rect. ABC'D'. It may be pointed out to the students that the base length and the altitude are equal in both the cases. Therefore area of ∥ gm, is nothing but Base × Altitude or Base × Height.

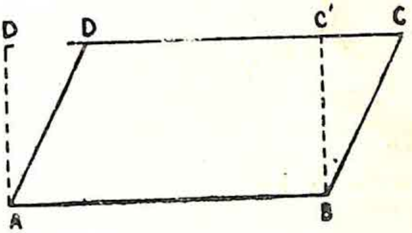


Fig. 26.

AREA OF CROSSROADS

Crossroads are quite common in actual life. Understanding of their area should invariably be obtained diagramatically. The clarity about the small shaded square coming twice is very important.

A few earlier problems should be solved with the help of the diagrams. This topic gives a very suitable material also for charts. These may be kept hanging in the class-room during the teaching of the topic. The roads may be shown by different colours. To give another experience, the cutting of 'road slips' may also be performed. The slip of the lengthwise road will be a continuous one and that of the breadthwise road will be broken into two pieces due to the non-availability of the central square piece second time. An extra piece

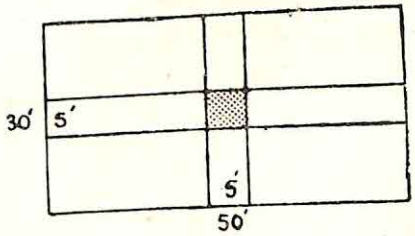


Fig. 27.

may be used to get this road also in a continued form. Naturally, the area of this central piece will have to be subtracted from the total area.

AREA OF CIRCULAR ROADS

Here also, the approach through the diagram is the best. The roads may be external or internal, there will always be two rectangles, one external and one internal. The general principle is : "the area of the road is equal to the difference of the areas of the two rectangles." It should be introduced as a general method for the area of all types of circular roads.

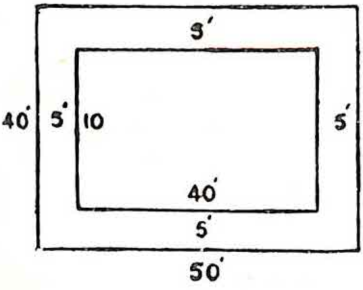


Fig. 28

AREA OF FOUR WALLS

The use of a cardboard model is a 'must' in this case. The opened model takes the shape of a rectangle. The four walls join up to make this rectangle. Let it be explained to the students that the length of this rectangle is nothing, but the perimeter of the floor of the room and that the breadth of the rectangle is the height of the room. The formula can therefore be given as :

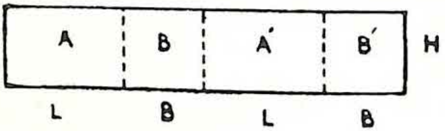


Fig. 29

Area of walls = Perimeter × Height

or $2(L+B) \times H$.

AREA OF A CIRCLE

This is the most difficult aspect of Area. The teacher will depend on the student's previous knowledge of the area of a ∥ gm and the circumference of a circle. A circle with radius 7 cm will be drawn. It will be cut into 20 parts or strips as shown in the diagram. These strips will be rearranged side by side as in the second figure. The resulting figure will be approximately a parallelogram. The teacher will first find out whether the students

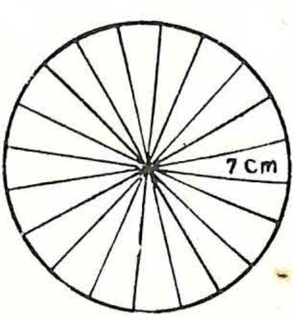


Fig. 30

remember the formula for the area of a ∥ gm. They need the measurements of base and the altitude of the figure. The requirements will be measured before them. The area of the resulting ∥ gm. is therefore = 22 × 7 = 154 sq. cm. The next question is how to connect this 22 cm. and 7 cm. with the dimensions of the original diagram. They are respectively half circumference and the radius of the given circle. After attempting one or two more cases in this illustrative manner, the students can be led to conclude Area of circle $= \pi r \times r = \pi r^2$.

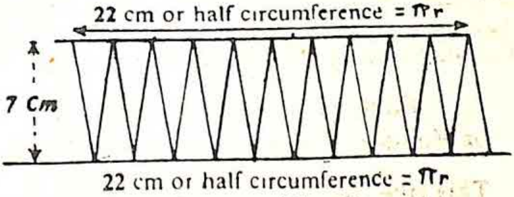

22 cm or half circumference = πr
7 Cm
22 cm or half circumference = πr

Fig. 31.

The teacher can arrive at this formula by some other method also. The students may be asked to draw circles on squared or graphed paper and then count the squares. This will give them areas in different cases. They can be helped to make a generalisation.

Similarly they may draw circles and circumscribe polygons; and then find the areas of the respective polygons. This will lead them to the idea that the area of the polygon is equal to $\frac{1}{2}r \times$ the perimeter of the polygon; and so to the idea that the area of the circle is equal to $\frac{1}{2}r$x the circumference of the circle.

VOLUME

What is Volume ? Perhaps, it will again be a mistake to be very particular about its definition. Some concrete illustrations can explain its significance more easily. Show a solid, a can, a box or a tin, and then say that the volume means 'capacity'.

Why do we learn about it ? The quantities of some articles of daily use are measured in terms of volume. The knowledge of volume helps in the understanding of different sciences.

The first thing the teacher has to do is to develop the idea of volume just as he would develop the idea of area. He may follow the same procedure, i.e.,

(i) choose a unit of volume, and

(ii) show how many times it is contained in the required volume.

The teacher will have to demonstrate the units in the form of 'cubic cm or 'cubic inch'. Then he will use a rectangular block of soap, say 9 cm by 4 cm by 3 cm. The block will be cut into cubic cm pieces, and thus 108 pieces will be obtained. The student should be directed to observe that in the bottom layer, on each square cm, there stands one cubic cm, therefore, there are as many cubic cms in the bottom layer as there are square cms in the area of the base, and the same is true of each layer. Hence they can be guided to the rule : "To find the number of cubic cms in a cuboid multiply the number of square cms in the base by the number of cms in the height".

$\therefore$ Volume of a cuboid $= L \times B \times H$.

or Volume of a cuboid = Area of the Base × Height.

This rule can be extended to the volumes of other uniform solids also. The construction of a cylinder can be shown by placing uniform circular layers one above the other. Thus its volume can be worked out by the multiplication of the area of the circular base with the height of the cylinder.

STOCKS AND SHARES

WHAT IS STOCK AND SHARE ?

The term 'stock' is used for the capital of a company. It is the initial value of total investment in that company. When the business is very large, a company consisting of a number of persons is formed. The stock of the company is subscribed by its members. These members are called share-holders. The capital is divided into a number of equal portions called shares.

Profits of the company are divided among the share-holders in proportion to the shares they possess. Each man's share of the profit is called 'Dividend'.

No share-holder can withdraw his money from the capital or stock. He can, however, sell the shares to someone else, and can also surrender to the purchaser his rights to the annual dividend.

If the company is prosperous, the value of the share rises; but if the concern is running at a loss, the value of the share goes down. The net capital of the company remains constant. This buying and selling does not affect this net capital.

If the company has fixed the value of a share at Rs 100 and is paying a handsome dividend, (say 5 on the capital or stock), the result is that a share may be selling at a higher price (say Rs. 110)

The entire situation can be summed up as follows for the convenience of the learner :

Rs 100 in stock = Rs 110 in cash
or Rs 110 in cash = Rs 100 in stock.
On a stock of Rs 100 the income is Rs 5
If income is Rs 5, stock is Rs 100.

If cash is Rs 110, income is Rs 5.
If income is Rs 5, cash is Rs 110

If these six connections are understood, the remaining work on stocks and shares is nothing but an application of the unitary method.

EXERCISES

1. Explain and illustrate the procedure of teaching the four fundamental rules of mathematics ?

2. What broader principles should be kept in mind while teaching arithmetic to the students ?

3. How will you introduce the following to the beginners :—

   (i) Metric measures.
   (ii) Discount.
   (iii) Area of a circle.
   (iv) Volume.
   (v) Proportion.
   (vi) Fractions.

4. How will you introduce and develop the idea of square root so that effective and complete comprehension takes place in the students ?

5. How can you relate the topics of fractions, percentage, unitary method, simple interest and discount with practical life ?

CHAPTER SEVENTEEN

THE TEACHING OF HOUSEHOLD ACCOUNTS

THIS BRANCH HAD been at one time meant for girl students only. It had been argued that since every girl had to run her household independently after marriage, she should be taught 'arithmetic of the household'. So, for girl students, Household Arithmetic was allowed as an alternative subject to algebra and geometry. Later on it was felt, that 'running a home' was the joint responsibility of the male and the female members. Earning and spending wisely depended on both of them and needed their mutual cooperation and understanding. In dealing with financial matters, the home should be conducted on sound business principles. Knowledge and practice of household arithmetic is a prerequisite for a 'happy home'. The emphasis on household arithmetic is expected to relate mathematics with actual life. It is also going to equip the learner with the mathematical knowledge of everyday use. The attempt is to bring arithmetic close to life. That will silence criticism that mathematics taught in schools is never used in life. It is aimed that the student leaves the school equipped with all the essential elements of mathematics. Household accounts is not to be treated as a separate branch; it is to be incorporated with the general syllabus of mathematics.

One fails to understand why household accounts should be started in the high school and not in earlier classes. It must be made an integral part of the syllabus of middle classes. This will make the syllabus of middle classes more utilitarian. At the same time there are in household accounts certain topics which are quite below the standard of a high school student and easily be understood by a student of middle classes. Certain topics of algebra and geometry, which don't fit in the middle school syllabus from the strictly utilitarian point of view, should be replaced by certain elementary aspects of household accounts. It will be useful from many points of view to introduce the household accounts in the middle classes. Even now, it has been given a place in these classes, but there is a need for greater share for this branch.

Should there be any option for the girls to take up Domestic

Arithmetic instead of Algebra and Geometry? The supporters of this option give the following arguments : (*i*) Algebra and Geometry will be little use to a girl when she becomes a housewife. (*ii*) Teaching of Domestic Arithmetic is a technical training to a girl for the successful running of the home. (*iii*) Knowledge of these accounts is badly needed for practical purposes in daily routine of women. The opponents put forth equally strong arguments : (*i*) Girls who aspire to prosecute higher studies in Mathematics and Sciences need schooling in Algebra and Geometry. (*ii*) Higher education in mathematics and sciences can no longer be denied to women because of the awakening in them and their resolve to move shoulder to shoulder with men in every walk of life.

The best way to solve this problem is to amalgamate household accounts with other parts of the syllabus of mathematics and adopt the same uniform syllabus for the boys and girls both.

The current trend is in favour of mathematics of practical value. It is reflected in the recent changes effected in the school syllabus. A few topics like taxes, insurance etc., have now been included in the middle school syllabus also. But more changes are needed to give Household Accounts its due place.

Some of the Western Schools have introduced the teaching of somewhat extreme form of socialized arithmetic. It has also been called 'business arithmetic', 'consumer arithmetic' and 'social economic mathematics'. Usually it is meant for those students who do not plan any further study in mathematics but who need additional training in home and business management. Experiments with this type of course are fairly recent, and only broad suggestion can be made about the contents of such a course. Here is an outline :

1. Measurements and computations; degree of accuracy, short ways of multiplying and dividing, etc.
2. Fundamental operations like fractions, decimals and percentages.
3. The simpler ideas of statistical methods; construction of various types of graphs; frequency tables; various types of averages and means; etc.
4. Index numbers. Their construction and use in connection with commodity prices, real wages, cost of living, business cycles, etc.
5. Household budgets. Cooperative enterprises.
6. Instalment buying.
7. Investments : stocks, bonds, mortgages; banking procedures, etc.

8. Insurance : home, fire, theft, property, accident, etc.
9. Taxation property, sales, income, direct and indirect; the cost of government.
10. Topics involving national policies, such as crop control, fixation of prices, social security, foreign exchange, distribution of national income, etc.

HOW TO TEACH HOUSEHOLD ACCOUNTS

It is taught best in the manner and through the situations in which it actually occurs in life. Financial management should be actually introduced in different activities of the students. Different activities may be started to provide for financial management. Kitchen, garden, knitting, cooperative shop, craft activities, cultural activities, celebration of days, planning a tour or a holiday, and the like should be invariably exploited to create situations for learning household accounts. Budgeting should be taught through budgeting, and account keeping through actual account keeping. The students should be given opportunities of preparing budgets and keeping accounts of shop, craft, household and school. The habit of keeping records of income and expenditure should be inculcated in them. By keeping accounts in a systematic way, one becomes economical.

The students should be given access to the school accounts registers. Cooperation of local business and industrial concerns should also be sought to show the students their cash books, day diaries, ledgers, etc. Local market situations should be utilised to the maximum possible to provide to the students real experiences in this field. Mere theoretical teaching of this branch is far from being enough. Only actual involvement in a purchase, can provide reliable answers to questions such as : What to buy : When to buy ? Where to buy ? and how to buy ? The most economical method of buying will automatically be learnt by them through actual experiences. Buying personally is decidedly economical though inconvenient.

Payments for the kitchen, cooperative and school purchases may also be made by associating some students with this activity turn by turn. The duty of maintaining and checking of the kitchen and cooperative store accounts should be entrusted to other groups also in their turn. The students should be advised to see whether it is beneficial to purchase things on cash, credit, instalment or hire purchase payments. They will not find any difficulty in understanding the difference between a cash memo, a bill and an invoice. They will understand easily the practical application of discount, sales tax, etc.

The opening, maintaining, transferring, operating or closing an account with a bank also can best be taught by organising a children's bank. Some students may open and operate different types of accounts at a post office. The writing or issuing different types of cheques may also be made a day to day experience for them. When they start their own small bank, or organise a cooperative store, they get an opportunity of investment. They become share-holders of an enterprise and look forward to some profit and expect some convenience.

The idea of insurance should be brought home to them by quoting examples from the locality.

Different agencies such as the post office, a bank, insurance offices, income tax offices, cooperative societies, business centres, small and big shops, etc., should be frequently visited by the students with specific purposes to understand their working and their connection with household accounts.

The teaching of household accounts will profit greatly, if students are addressed by post masters, bankers, bank managers, insurance agents and officers, income tax officers, cooperative inspectors, businessmen, shopkeepers, etc. Being experts in their trades, these persons will be in a position to give authentic information about them.

The school should acquire a sufficient collection of registers and forms connected with this branch e.g., day books; cash books; ledgers; bill books; cash memos; budgets—family as well as state; pass books; different forms prevalent in post offices and banks; cheque books; licences.

EXERCISES

1. Discuss the place and importance of household accounts as a part of the syllabus of mathematics.

2. When and how should household accounts be taught to the students ?

3. Illustrate the methods and techniques which you will try to use for the teaching of household accounts ?

CHAPTER EIGHTEEN

THE TEACHING OF ALGEBRA

WHAT IS ALGEBRA ?

THE WORD 'ALGEBRA' is Arabic in origin. It is a corruption of the word 'Al-jabr'al-muqabulah' where 'al' means 'the', 'jabr' refers to the operation of transferring a quantity from one side of an equation to another (of course, after performing a change in its sign) while 'muqabulah' means the process of subtracting similar quantities from both sides of an equation.

It is a generalisation of arithmetic. In teaching and learning, its principles are frequently needed to return to corresponding situations in arithmetic. With the introduction of algebra, the student learns to extend the number system of arithmetic so that the four fundamental operations can be performed in all cases. The first extension is the introduction of the negative numbers, to provide for the subtraction of larger from smaller numbers.

It is also thought to be the most difficult and abstract of all the branches of mathematics. But it is primarily taught for manipulative skill, solution of problems by equations, a power of generalisation and use of formulae, and idea of functionality.

As it has been called a generalized arithmetic, so it may be related to geometry by saying that algebra is only written geometry and geometry is merely pictured algebra.

WHY IS ALGEBRA TAUGHT

Algebra is taught because—

1. It is useful in other branches of mathematics. It has especially simplified for the learner, many problems of arithmetic.
2. It gives compact formulae or generalisations to be used in all cases. The solution of problems by equations and by factorisation are examples of this.
3. It has a practical value in many of the trades and industries.

4. It provides an effective way for expressing complicated relations.
5. It gives a new, good approach to the study of abstract mathematical relationships through the use of a new language and a new symbolism.
6. It inculcates the power of analysis.
7. Verification of results is simpler and more satisfactory in algebra than in any other branch of mathematics. It develops confidence among the students.
8. It helps in the generalisation of scientific truths into simple and compact formulae.
9. It is a good instrument for mental training.

TEACHING OF DIFFERENT TOPICS OF ALGEBRA

THE SIGNED NUMBERS

The first need is to attach certain meanings to the signs + (plus) and — (minus).

1. Kind of operation : + add; — subtract.
2. Direction : + rightward, upward ; — leftward, downward (opposite directions).
3. Directed or signed numbers : + positive; — negative.
4. Effect on direction : + continuation in direction already established; — reversal of such direction.

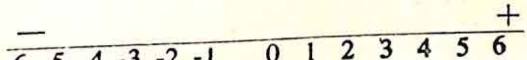


Fig. 31

A number scale is used to represent directed numbers by making a start from an arbitrary zero point. It is agreed that a number having no prefixed sign be considered positive. It is now necessary to make rules for operating with negative numbers.

Addition:—The following are the four possible cases :

$$\begin{array}{cccc} \text{(a)}\ +5 & \text{(b)}\ +5 & \text{(c)}\ -5 & \text{(d)}\ -5 \\ +4 & -4 & +4 & -4 \\ \hline \end{array}$$

Case (a) means that starting from the zero point 5 units are counted to the right, then 4 more units to the right, ending at +9. The teacher may explain like this : "We can interpret + as something in our possession. A person has Rs 5=00 in one pocket and Rs 4=00 in the other. The total amount possessed by him is naturaly Rs 9=00."

Case (b) shows that 5 units are counted to the right of zero, then from +5, 4 units are counted to the left, ending at +1. The teacher may explain, like this :—"We can say that a person possesses Rs 5=00 and owes Rs 4=00. The net amount possessed by him is naturally Re 1=00."

Case (c) states that 5 units are counted to the left of zero, then from —5, 4 units are counted to the right, ending at—1. The teacher may explain like this :—"We can say that a person owes Rs 5=00 and possesses Rs 4=00. His net debt will be Re 1=00."

Case (d) states that 5 units are counted to the left of zero, then 4 more units to the left, ending at—9. Or it can be said that a person owes Rs 5=00 to A and Rs 4=00 to B. His net debt will be Rs 9=00.

With a few more examples of this type, the rule can be framed, namely "to add two numbers with like signs, prefix the common sign to their numerical sum; otherwise prefix the sign of the numerically larger number to their numerical difference."

SUBTRACTION

The teacher announces to the class, "Let us perform the operation of subtraction on the four cases in hand."

Case (a) is a simple problem of taking out 4 from 5.

Case (b) indicates the subtraction of —4 from +5. When a person's debt is written off, he gets richer by that amount. In this case a debt of Rs 4=00 is written off. So the person who already possesses Rs 5=00, gets richer by Rs 4=00. He has been saved from a debt of Rs 4=00. His net possession increases by four rupees. Therefore subtracting —4 means adding +4.

Case (c) states the subtraction of +4 from —5. A person's possession is —5 rupees. Another sum of rupees +4 is to be taken out from his possession. He becomes poorer by four rupees in this process. Therefore his net possession is —9 rupees. Thereby subtracting +4 means adding —4.

Case (d) states the subtraction of —4 from —5. A person's possession is —5 rupees. The subtraction of —4 means that a debt of 4 rupees is to be written off. When a debt is written off, the debtor indirectly gains. He becomes richer by four rupees. Therefore his net possession is —1 rupee. Subtraction of —4 means addition of +4. The ground is prepared to announce the general rule for subtraction, namely, change the sign of the subtrahend and proceed as in addition.

MULTIPLICATION

In arithmetic, it has been found in an earlier chapter that multiplication is repeated addition of like numbers. The case of multiplying +4 with+3 is not difficult to explain. The second case is that of multiplying —4 with +3. This can be interpreted to mean that —4 has to be added three times, which gives the product as —12. The third case is that of multiplying +4 with —3. One knows that in a multiplication of two numbers, either of them can be made the multiplier. This problem can be interpreted to mean that —3 has to be added four times. That gives the product as —12. For making the fourth case of the multiplication understandable, the example given below may be found useful. It helps in understanding the other three cases also.

There are fifty students on the rolls of a class and the school charges from forty of them Rs 5=00 as tuition fee per head. The students being on the rolls can be regarded as +ve; and as they are fee-paying students they are a source of income for the school. The product of +40 and +5 Rs is +200 rupees. Suppose 10 fee-paying students leave the school. They become—ve. The school loses Rs 50=00 by their leaving. The product of —10 and +5 Rs is —50 rupees. Further suppose that because of their poverty the remaining ten students of the class get a scholarship of Rs 10 per head from the school. The school loses Rs 100=00 on them at Rs 10=00 each. The product of +10 and —Rs 10 is —100 rupees. If five out of these ten scholarship cases leave the school, they become negative. Their income for the school was already negative. By their departure the school saves or gains Rs 50=00. Therefore the product of —5 and —10 is +50 rupees.

Another example, given below, can be used to elucidate the multiplication of directed numbers:—A owes Rs 10=00 to each of the four persons. Therefore the net balance of his possession is —40 rupees i.e., if you multiply —10 and +4 you get —40. If two of the creditors die, two of the debts are automatically written off. The two creditors on account of their death become negative. By their death 'A' is saved from the payment of twenty rupees. The net balance of his possession will increase by this amount. The product of —10 and —2 is therefore +20.

These results indicate the rule for the multiplication of two signed numbers : the product of two numbers with like signs is positive; with unlike signs it is negative.

DIVISION

The rule for the division of one signed number by another follows directly from multiplication. In the straight problems of division, it

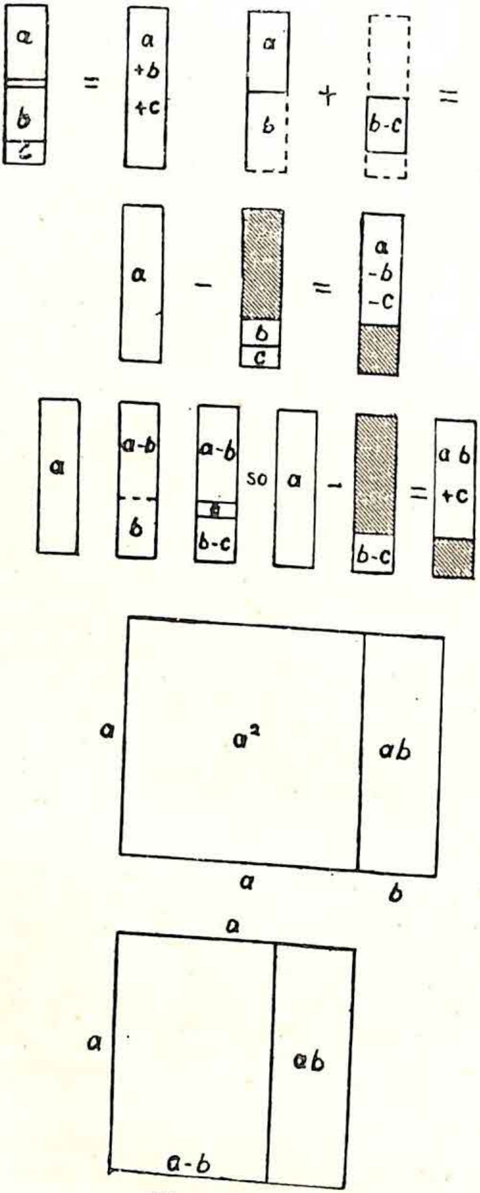


Fig. 32 (a), 32 (b)

is known that Divisor × Quotient = Dividend. If the Dividend is positive, Divisor and Quotient will have like signs. If it is negative, the two factors will have unlike signs.

THE USE OF BRACKETS

A pair of brackets is a sort of little box containing some quantities which are to be regarded all together as one quantity. Different cases of the use of the brackets are illustrated below:--

(1) $a+(b+c)=a+b+c$

(2) $a+(b-c)=a+b-c$

(3) $a-(b+c)=a-b-c$

(4) $a-(b-c)=a-b+c$

(5) $a(a+b)=a^2+ab$

(6) $a(a-b)=a^2-ab$

FIRST DEGREE EQUATIONS

The procedure with the first degree equation is direct. All the terms containing the unknown are removed to one side (the left by convention) : and all constant terms to the other side of the equality sign. The terms on each side are then collected, and division by the coefficient of the unknown gives its value. This value satisfies the equation.

A few simple equations can be borrowed from arithmetic and geometry also : such as

A (area) = L (length) × B (breadth)

$$\text{I (Simple interest)} = \frac{\text{P(principal)} \times \text{R(rate)} \times \text{T(time)}}{100}$$

V (volume) = L(length) × B(breadth) × H(heigh).

$\triangle$ (area of a triangle) $= \frac{1}{2} \times$ B(base) × H (height).

In a $\triangle$, $\angle A + \angle B + \angle C = 180^0$.

In a rt. angled $\triangle$ $(\text{Hyp})^2 = (\text{Base})^2 + (\text{Perp.})^2$

The idea of equation can be easily introduced with the help of examples of the following type :—

4 books + 3 books = 7 books or $4b+3b=7b$

8 chairs + 4 chairs = 12 chairs or $8c+4c=12c$

The problem that the cost of 7 books is Rs 22.40 paise, find the cost of one book, can be taken as 'equation problem.'

$$7b=22.40$$

$$b=\frac{22.40}{7}=3.20$$

Now the teacher should extend the problem and say, "If five rupees were added to the cost of 7 books the sum is Rs 27.40 paise. Find the cost of one book."

It can be sloved as :

$$7b+5=27.40$$

But, how to get rid of '5' so that only '$7b$' is left on one side ? The teacher will take the help of simpler examples, such as :—

$$12=12$$

$$12-2=12-2 \quad i\,e. \quad 10=10$$

$$12-5=12-5 \quad i\,e. \quad 7=7$$

That is, any quantity can be subtracted from one side of the equation, provided the same is subtracted from the other side also. When a shopkeeper weighs 8 Kgs of wheat flour, he has 8 Kgs weight on one side and 8 Kgs of wheat flour on the other. But if the customer recollects that he needs only 5 Kgs, the shopkeeper has to take out 3 Kgs weight from one side and the same weight of wheat flour from the other side, to balance the two.

Reverting to the problem proper, the teacher will say,

$$7b+5-5=27.40-5 \quad \text{or} \quad 7b=27.40-5 \quad \ldots(i)$$

$$\text{or} \quad 7b=22.40 \quad \text{(the same as before)}$$

The teacher reshapes the problem as, "If five rupees were subtracted from the cost of 7 books, the remaining amount is Rs 17.40 paise. Find the cost of one book ?

He proceeds as usual :—

$$7b-5=17.40$$

The earlier examples may be used in a modified form to explain that here 5 is to be added on both sides.

$$7b-5+5=17.40+5 \quad \text{or} \quad 7b=17.40+5 \quad \ldots(ii)$$

$$\text{or} \quad 7b=22.40$$

The equations (i) and (ii) can be used to arrive at the generalisation 'any quantity can be shifted from one side of the equation to the other by changing its sign.'

The students can easily solve

$7x=28$ and say $x=4$.

The teacher asks, "What is the rule about it ?" They answer, 'A numerator when shifted from one side to the other, becomes denominator.' Similarly it can be further generalised that 'a denominator when shifted from one side of the equation to the other, becomes numerator.'

The four axioms involved in the solution of linear equations can be written in another form also :—

1. If equal quantities are added to equal quantities, the results are equal.

2. If equal quantities are subtracted from equal quantities, the results are equal.

3. If equal quantities are multiplied by equal quantities, the results are equal.

4. If equal quantities are divided by equal quantities, the results are equal.

GRAPHS

WHY TEACH GRAPHS

The invention of graphs is due to Descartes. Graphic representations have become very popular today. The progress of an industry, the rise in population, the improvement of the employment situation, the course of an illness, all these and many more can be brought to a clear picture by means of graphs. The knowledge of graphs forms a necessary part of general knowledge. It gives a power of dealing with questions that are well-nigh impossible of solution or are too difficult to solve by ordinary algebraical methods. It also gives the student an excellent opportunity for criticising his own work : a graph that suddenly moves off in an unexpected direction or a point that spoils the smoothness of a curve soon leads the pupil to suspect a mistake. It is a very concise way of representing a statement. It is helpful in the study of different sciences. The knowledge of graphs is an essential foundation for higher mathematics. Graphs can be used to create interest even in abstract lessons. A graphic representation appeals to the aesthetic sense. The primary use of a graph is to exhibit to the eye a series of simultaneous values of two quantities. It makes an immediate appeal to the eye.

SOME POPULAR TYPES OF GRAPHS

Bar, column, circle and graph paper are the different means to depict statistical relations. Such graphical devices are being increasingly and prominently used in newspapers, magazines, and other current publications.

AREA OF 1000 ACRES

Residential Area	Roads	Waste Land	Canal Irrigated Area	Well Irrigated Area	Barren Land
2%	5%	19%	28%	26%	20%

Fig. 33. (a)

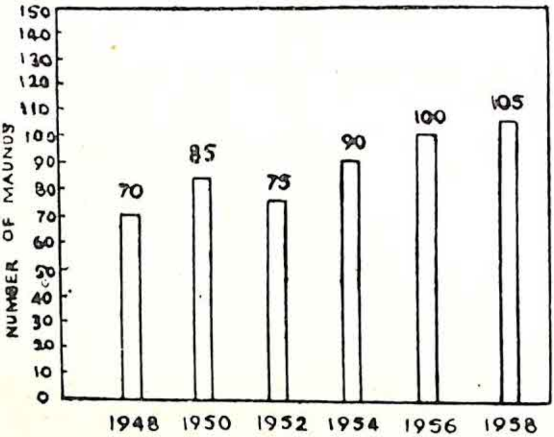


Fig. 33. (b)[2]

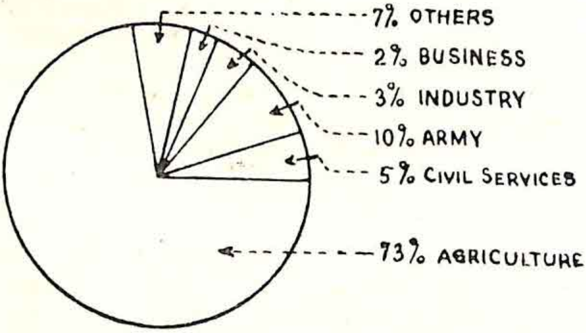


Fig. 33 (c)

Graphed paper graph of amounts in simple and compound interest rules.

Amount of Rs. 100 at 10% compounded annually.

The graph of the amount at simple interest follows a straight line, whereas that of the amount at compound interest follows a curve joining 100, 110, 121, 133.1, 146.4, 161, 177 and so on.

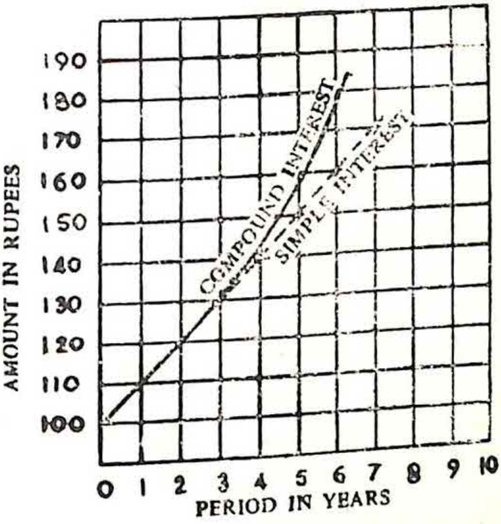


Fig. 34

THE TEACHING OF GRAPHS

The first stage in teaching of graphs should include graphing of statistics. This has really nothing to do with algebra, but there are many lessons to be learnt from such work. It is essential that attention should be paid towards the reading and drawing of graphs from purely arithmetical data before any algebraic notation is introduced. The reading and drawing of graphs necessitate the use of the graphed blackboard Graphical work takes a lot of time, and it is useful to preserve the graphs once drawn for future reference. Hence it is advisable to work in note books ruled for the purpose.

After a good practice in reading the graphs, comes the stage of plotting or drawing them.

It should be impressed on the students that a graph represents relationship between two quantities, one of which is measured along the horizontal line and the other along the vertical line of the graph paper. A chart showing the +ve and −ve directions along both the horizontal and vertical lines should remain hanging in the class-room throughout the teaching of graphs.

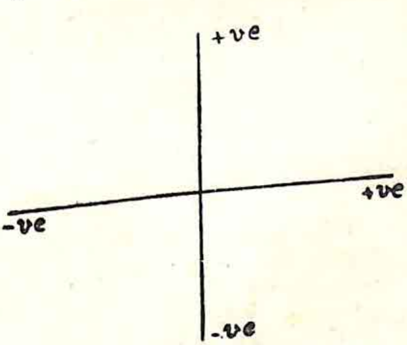


Fig. 35.

Practice in plotting should be started from the plotting of points. The students may be asked to plot some numbers and their squares.

[(1, 1); (—1, 1); (2, 4); (—2, 4); (3, 9); (—3, 9) etc.] or the points representing the cost prices of one to six books e.g., when one book costs Rs 1.50. Different points to be plotted in this case will be [(1, 1.50); (2, 3); (3, 4.50) and so on]. In the latter case, the teacher will ask the students to join the three plotted points. A straight line will be the result. The graph of two proportional variables is a straight line. Thus the graph of the costs of different quantities of a commodity will be a straight line. Similarly the graph of the motion of a person travelling 3 miles per hour will be a straight line. A good deal of practice in plotting the points such as —1, 2); (—2, —3); (2, —2); (—3, —4); (—2.5, 1.5) should be given for a considerable period of time.

ALGEBRAICAL GRAPHS

The two axes on the graph are named as *x*-axis and *y*-axis, their meeting point being (o, o). The start in this aspect of graphs may be made from the drawing of the graphs of relations or equations like $y=2x$ or $y=x+1$.

In either of these equations, for every value of x, there is a corresponding value of y. The corresponding values of the two variables are calculated thus : $y=2x$.

X	1	2	3	4	-1	-2	-3
Y	2	4	6	8	-2	-4	-6

Fig. 36. (a)

But three values are enough in such cases.
Similarly in case of $y=x+1$.

X	1	2	-1	-2	-3
Y	2	3	0	-1	4

Fig. 36. (b)

The teacher will emphasise that such algebraic equations, when drawn on a graph, are represented by straight lines. The accuracy of the drawing can be easily verified in these cases. If the joining of different plotted points gives a straight line, the drawing is accurate, otherwise not. Moreover, any other point that falls on that line must satisfy the given equation.

SOLVING OF LINEAR EQUATIONS

The five commonly used methods of solving simultaneous equations are (1) the method of elimination by comparison, (2) the method of elimination by addition and subtraction, (3) the method of elimination by substitution, (4) the graphical method, and (5) the method of determinants.

The graphical method can be used, of course, only with students who have previously studied the meaning and construction of graphs of linear equations. Its principal advantages lie in the fact that it is interesting and that it illustrates things in a very convincing manner.

Its main disadvantages are that it provides only approximate solutions instead of exact ones, which often introduces apparent discrepancies and gives the student a consequent feeling of dissatisfaction. Secondly, it is a comparatively slow, tedious, and inefficient method for solving simple linear equations. For this reason, after students have learnt more exact and efficient algebraic methods, they are likely to prefer them to the graphical method.

The solving of two linear equations, will involve the drawing of the straight lines representing the two separately. Their point of intersection is the point or value which lies on both the lines and hence satisfies both. Thus the point of intersection gives the solution of the two equations.

POINTS TO BE KEPT IN MIND WHILE TEACHING GRAPHS :

1. Graphed black-board must be provided in the school.
2. Graphs of the progress of the school should be got prepared by the students.
3. Graphs should not only be drawn but also interpreted.
4. Students should know that the graph is a straight line through the origin if the two quantities involved are proportional.

FACTORISING AND FORMULAE

When an expression is a product of two or more expressions, each of the latter expressions is called a factor.

Thus $x^2+(a+b)x+ab$ is the product of $(x+a)$ and $(x+b)$

$\therefore$ $(x+a)$ and $(x+b)$ are the factors of the expression $x^2+(a+b)x+ab$.

The process of finding the factors of an expression is called factorising.

Factorising can be taught logically, and connected with the previous knowledge of multiplication and division in the following manner :—

Multiplication	Division	Factorising
$5\times4=?$	$20\div5=?$	$20=?\times?$
$8\times6=?$	$48\div8=?$	$48=?\times?$
$a(a+b)=?$	$(a^2+ab)\div\mathrm{a}=?$	$a^2+ab=?\times?$

The teachers should not insist upon doing very intricate problems on factorising. This topic is one of the difficult topics of algebra. It should be presented in an interesting manner. Only useful cases of factorising should be taught.

A keen sense of observation and analysis pay the student most in factorising. The first thing is to see whether there is a factor which is quite evident. Thus in $8a^2-16ab$, $8a$ should at once come to the child's mind. The other factor is obtainable by dividing each term by $8a$.

It should not at all be difficult to deal with expressions like $3a(a+b)-b(a+b)$. Here $(a+b)$, in a way, shouts at the learner. But sometimes the student may get confused in such straight cases also. He may multiply $3a$, $\mathrm{a}+b$; and b, $a+b$; get the expression as $3a^2+2ab-b^2$, and then start factorising it. This shows that the teacher should lay stress on the evident and "shouting" type of factors.

The teacher will also have to create among the students a sense of grouping the connected terms. A useful tip here is to spot some letter that occurs only in two terms and to group those terms together. Thus, in $ab+cd+ac+bd$, a occurs in two terms; and those terms give us $a(b+c)$ as sub-factors; the other two terms give us $d(b+c)$. Therefore the expression equals $(b+c)(a+d)$.

The difference of two squares gives an opportunity of going into geometry. The geometric illustration shows clearly why the area represented by the product $(a+b)$ $(a-b)$ is the same as that represented by a^2-b^2. Now suppose a is 8 and b is 2. Then $a^2-b^2=64-4=60$. But $(a+b)=10$ and $(a-b)=6$ and (10) $(6)=60$. Thus may also be illustrated arithmetically the fact that $a^2-b^2=(a+b)$ $(a-b)$. The other type of forms may easily be illustrated in the same manner. These types of forms must be learnt thoroughly. They must be understood and also memorised.

THE TWO FUNDAMENTAL REQUISITES ARE :

1. Knowledge of the type forms of the various special kinds of factors.

2. Ability to recognise, identify, and associate particular cases and particular elements with the general types to which they belong.

For example, the factorisation of x^6-y^6 involves its identification and association with two particular cases of the difference of two squares, and the difference of two cubes.

The students should also be given the opportunity of factorising the forms like $5+6x+x^2$ and $90+93x+25x^2+2x^3$. The occurrence of terms like x^2 and $2x^3$ in the first place should not be a rule.

They should be encouraged to enjoy the symmetry of problems like $(a-b)^2+(b-c)^2+(c-a)^2$, $a^2(b-c)+b^2(c-a)+c^2(a-b)$, $x(y-z)^3+y(z-x)^3+z(x-y)^3$. They should even be asked to invent some such problems.

SOME METHODS OF FACTORISATION

1. METHOD OF FINDING THE SUM AND THE PRODUCT OF THE ROOTS

Example $x^2-11x+28$.
In the quadratic equation $x^2-11x+28=0$
the product of the roots$=28$,
and sum of the roots$=-11$
Therefore the roots are -7 and $-$
Hence the factors of $x^2-11x+28$ are $(x-7)(x-4)$

2. EXPRESSING THE QUADRATIC AS A DIFFERENCE OF TWO SQUARES

This method is applicable to problems like ax^2+bx+c,

When a or c is a perfect square and b is an even number. (It need not be applicable to all such cases).

Example :$-5x^2+16x-16$

$$\begin{aligned}9x^2-(4x^2-16x+16)&=9x^2-(2x-4)^2\\&=(3x+2x--4)\,(3x-2x+4)\\&=(5x-4)\,(x+4).\end{aligned}$$

3. BY PUTTING THE VALUE OF THE VARIABLE

This is also, in a way, a method of trial. If by putting a certain value of the variable the quadratic becomes zero, the variable reduced by this value is a factor.

Example :—Factorise $x^3+9x^2+8x-60$.

Putting $x=\pm1$, does not fulfil the condition, but $x=2$ makes the quadratic zero, therefore $x-2$ is a factor, and so on.

DIVISION BY ZERO

The impossibility of dividing by zero should be shown and brought home to every student of mathematics generally and to every student of algebra particularly. Many fallacies arise through inadvertent division by zero. Such statements as $1=2$, or any number is equal to any other number, may be obtained in this manner. For example :

Let $x=y=3$	$\therefore\ 15x=15y$
	and $10x=10y$
By subtraction	$15x-10x=15y-10y$
By transposition	$15x-15y=10x-10y$
Or	$15(x-y)=10(x-y)$
$x-y$ is common	
$\therefore\ 15=10$	

Note that here the common factor $x-y$ has been cancelled from both the sides. This has been done according to the rules of division. Both the sides, in fact, have been divided by $x-y$, which is equal to zero. The students can, of course, astonish others with this magic of mathematics but they themselves should never fall into the trap. Let it be known to them, that division by zero is not permissible.

Similar two common errors are generally made by students in solving problems. One is given in the following example :

$$x^2-9=x^2-5x+6$$

or $$(x-3)(x+3)=(x-3)(x-2)$$

The factor $x-3$ is frequently removed, with the impossible result $x+3=x-2$. The proper root $x=3$ is thus lost by not transposing and collecting terms.

Another error more frequently made is shown in solving for x the equation $ax+b^2=bx+a^2$

$$(a-b)x=a^2-b^2$$

$x=a+b$, if and only if $a-b\neq0$. This last condition is vitally necessary, for if $a=b$, the given equation is an identity, which is true for any value of x.

FORMULAE

A formula is a form of equality which holds true for all values of the variable. A formula is a shorthand statement.

A thorough knowledge of the important formulae and their applications to various algebraic operations is absolutely necessary. They have to be intelligently memorised. A chart of these may be prepared and always kept hanging in the mathematics class-room. Their ready knowledge is of utmost importance. The following are some of the important formulae of everyday use :—

$$(a\pm b)^2=a^2\pm 2ab+b^2.$$
$$(a+b)(a-b)=a^2-b^2.$$
$$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$$
$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$
$$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$$
$$a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$a^3-b^3=(a-b)(a^2+ab+b^2).$$

Their understanding involves only simple multiplication and division. The teacher may test the student's knowledge of these formulae on any day and without prior notice. Construction of these formulae involves real ability of the students. The reconstruction of formulae is more important than their retention.

There will also be opportunities to verify and apply these formulae.

INDICES

The Index Notation is a simple and brief representation of the multiplication of the same number. Area of square and volume of cube are good cases for the application of indices.

The following examples will make the idea clear :—

$2\times 2=4$ (It is the area of a square with side 2 inches)
It can be called the 'square of 2' or 2^2.

$2\times 2\times 2=8$. It is the volume of cube with side 2 inches. It can be marked as the 'cube of 2' or 2^3. That is, if you convert 2×2 and $2\times 2\times 2$ into index form, you get 2^2 and 2^3 respectively.

Similarly, the teacher can ask the students to convert $2\times 2\times 2\times 2$, $a\times a\times a\times a\times a$ and $x\times x\times x\times x\times x$ into their respective index forms. Conversely, they can be guided to notice that a^8 means 'a multiplied by itself eight times' x^{11} means 'x multiplied by itself eleven times' and a^m means 'a multiplied by itself m times'. How to write a^m as a continued product will also be explained as $a^m=a\times a\times a$...... ......m times.

THEIR MULTIPLICATION

How to multiply terms given in index forms is an important and interesting thing to learn. The beginning will be made by a little more concrete cases; such as—

$2^2 \times 2^3 = 4 \times 8 = 32$

But it is also $= 2 \times 2 \times 2 \times 2 \times 2$

(i.e., 2 multiplied by itself 5 times).

$= 2^5 = 32$

$\therefore \quad 2^2 \times 2^3 = 2^5$

Similarly $3^3 \times 3^4 = 27 \times 81 = 2187$

But it is also $= 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$

(i.e., 3 multiplied by itself seven times)

$= 3^7 = 2187$

$\therefore \quad 3^3 \times 3^4 = 3^7$

Similarly $a^5 \times a^6 = a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a.$

(i.e., a multiplied by itself eleven times)

$= a^{11}.$

In this manner, the generalisation can be arrived at by writing

$a^m \times a^n = (a \times a \times a \ldots\ldots\ldots m \text{ times})\ (a \times a \times a \ldots\ldots\ldots\ldots n \text{ times})$

$= a \times a \times a \ldots\ldots\ldots (m+n) \text{ times}$

$= a^{m+n}.$

So, if the base is the same, the index of the power of the product is the sum of the indices of the powers.

Cases like $a^2 \times a^3 \times a^4$ and $a^x \times a^y \times a^z$ will be taken for the purpose of practice.

THEIR DIVISION

This will also be introduced through concrete cases; such as—

$$\frac{2^5}{2^3} = \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = 2 \times 2 = 2^2 = 4$$

$$\frac{3^7}{3^4} = \frac{3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3} = 3 \times 3 \times 3 = 3^3 = 27$$

$$\frac{a^6}{a^4} = \frac{a \times a \times a \times a \times a \times a}{a \times a \times a \times a} = a \times a = a^2$$

These cases will show that in division, if the base is the same, th index of the power of the quotient is obtained by subtracting th power of the divisor from that of the dividend.

To make the necessity of 'same base' clear, the teacher will as the students to try the multiplication of a^5 with b^3 and the division o a^{10} by b^4

THE MEANING OF ZERO POWER

This can be explained by both the methods of division and mu tiplication.

By now the students will have known that

$\frac{a^m}{a^m} = a^{m-m} = a^0$ They also will be knowing tha

$a^m \div a^m = 1$.

Also $a^n \times a^0 = a^{n+0} = a^n$. The only possibility in this case also is that $a^0 = 1$. The generalisation, therefore, will be that "any number raised to the power 0 is 1, except 0^0 which cannot be determined."

THE POWER OF POWER CASES

In this case also concrete cases will be the best beginning; such as—

$(2^3)^2 = (8)^2 = 64$ which is also $= 2 \times 2 \times 2 \ldots\ldots$ 6 times,

$\therefore \quad (2^3)^2 = 2^6$.

Similarly :—

$(3^2)^4 = (9)^4 = 6561$ which is also $= 3 \times 3 \times 3 \ldots\ldots$ 8 times.

$\therefore \quad (3^2)^4 = 3^8$

Similarly :—

$(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3 = a^{3+3+3+3} = a^{12}$.

Hence $(a^m)^n = a^{mn}$.

The following rule can be framed :—

In taking power of some quantity containing a power of a base keep the base the same and multiply the indices of the powers.

QUADRATIC EQUATIONS

There are at last two valuable facts about quadratic equa tions :—

(i) There is some novelty about solving an equation whicl contains x^2. (ii) There is certainly novelty in finding that an equatio can be true for more than one value of x.

METHODS OF SOLVING

There are in all five methods of solving quadratic equations :—

1. Graphical method ; 2 Inspection method (in the case of incomplete quadratics); (3) Factorising; (4) Completing the square; and (5) Quadratic formula.

THE GRAPHICAL METHOD

It is not strictly an algebraic method and can give only approximate solutions; but it is very valuable in clarifying the meaning and nature of the two roots, just as the graph itself is useful in bringing out certain important characteristics of the function. There are four essential things to be brought home to the students : (1) the graph is the geometric representation of the quadratic equation and as the independent variable takes on different values, the function also takes on different values; (2) since the general form of the quadratic equation is $ax^2+bx+c=0$, seek to find those values of x which will make the function $y=ax^2+bx+c$ have the value zero; (3) the only points on the graph for which the function can have the value zero are those points which are on the x—axis; and (4) consequently the abscissas of those points give the values of the independent variable which satisfy the equation. The graph also can be used effectively to show why two real and distinct roots may exist, why the roots are sometimes real and equal, and why, in certain cases, there are no real roots.

Because it is slow and tedious, the graphical solution should not be used for many problems. Its purpose will have been served when it has been used in a few cases for clarification. The major part of the work with quadratic equations should be concerned with algebraic rather than graphical solutions.

METHOD OF INSPECTION

It is a very simple and obvious method. For example—The observation and inspection of $x^2-5x+6=0$ can lead the student to find two numbers which when added give -5 and when multiplied give $+6$. The number $+6$ can be a product of only two groups of —ve numbers -1, -6; and -2, -3. The summing up of the numbers of the second group fulfils the second condition. Therefore, $x=2$ and 3 are put in the equation; the student will be further satisfied about its roots.

METHOD OF FACTORISING

The students already know about factors. This method does not pose any new difficulty; it will only require a little careful attention.

There is one point which should be specially clarified by the teacher. When an equation such as $x^2-5x+6=0$ is given in factorised form as $(x-2)(x-3)=0$, it is not always clear to the students why one has the right to set the factors separately equal to zero and thus to get two linear equations. It should be explained that there are two possibilities of the satisfaction of the equation. If either of the factors is zero the equation will be satisfied; and the only way in which the equation can be satisfied is that at least one of the factors should be equal to zero. Moreover, the roots of the equation are given in the form $x=2$. This method is also not to be continued for too long, because of 3. Most of the time is to be reserved for the general method to come. In a few examples in the beginning, the results should be verified and checked by this method.

METHOD OF COMPLETING THE SQUARE

Before adopting this method, the teacher should make sure that the students know thoroughly that $(a+b)^2=a^2+2ab+b^2$ and $(a-b)^2=a^2-2ab+b^2$, and also can apply these. To be sure about their previous knowledge, he can ask the class to fill up the gaps in such cases as

$$x^2+4x+4=(\qquad)^2,\quad x^2-\qquad+25=(\qquad)^2$$
$$x^2-4x+\quad=(\qquad)^2,\text{ and } 4x^2+12x+\qquad=(\qquad)^2.$$

First, the teacher may take such an example as $x^2=16$. The class will at once suggest $x=4$ as a solution, but let him also solve the equation by factors and get $(x+4)(x-4)=0$, $x=4$ or -4. Testing can assure him that 4 and -4 are both solutions.

Then the teacher will set the problem $x^2+6x=27$.

It will be treated as follows :—

$$x^2+6x+\ldots\ldots\ldots=(x+\ldots\ldots)^2$$

Or, by completing the square,

$$x^2+6x+9=(x+3)^2$$
$$\therefore\ x^2+6x+9=27+9$$

Or $(x+3)^2=36$, which type is already known to the students. They can catch the point that

$$x+3=6 \text{ or } -6$$
$$\therefore\ x=3 \text{ or } -9$$

It will not be necessary, however, to spend much time in actually solving equations by this method, since its principal function here is to provide a means for developing the general quadratic formula.

Method of Quadratic Formula

This formula is very important and should be mastered by every student. Its development requires the use of the method of completing the square. The development should be carefully explained, and the students should be tested for their understanding of every step.

The equation of the type $x^2-x-3=0$ will be given to the students for solution. It will not be possible to solve it with the method of factorising. The students will be asked to try the method of completing the square, namely.

$$x^2-x-3=0$$

$$\text{or}\quad x^2-x\quad=3$$

Making L.H.S. a perfect square,

$$x^2-x+\frac{1}{4}=3+\frac{1}{4}$$

$$\text{or}\quad\left(x-\frac{1}{2}=\right)^2\frac{13}{4}$$

Or, taking square root on both sides :—

$$x-\tfrac{1}{2}=\pm\frac{\sqrt{13}}{2}$$

$$\text{or}\qquad x=\tfrac{1}{2}\pm\frac{\sqrt{13}}{2}=\frac{1+\sqrt{13}}{2}$$

Another example, $2x^2-3x+4=0$, will also be solved in the same manner—

$$2x^2-3x=-4$$

$$\text{or } x^2-\frac{3}{2}x=-2$$

Making L.H.S. a perfect square,

$$x^2-\frac{3}{2}x+\frac{9}{16}=-2+\frac{9}{16}$$

$$\left(x-\frac{3}{4}\right)^2=-\frac{23}{16}$$

Taking square root on both sides,

$$x-\frac{3}{4}=\pm\frac{\sqrt{-23}}{4}$$

or $$x = \frac{3}{4} \pm \frac{\sqrt{-23}}{4}$$

$$\therefore \quad x = \frac{3 \pm \sqrt{-23}}{4}$$

To enable the students to arrive at the generalisation, the teacher will ask them to solve $ax^2+bx+c=0$ in the same manner.

$$ax^2+bx+c=0$$

or $$x^2 + \frac{b}{a}x = \frac{-c}{a}$$

Making L.H.S. a perfect square,

$$x^2+\frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a}+\frac{b^2}{4a^2}.$$

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = \frac{-4ac+b^2}{4a^2}.$$

Taking square root,

$$x+ \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

or $$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

The teacher will go back to the previously solved numerical problems to enable the students to understand that $\frac{1 \pm \sqrt{13}}{2}$ is nothing but $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ of the equation $x^2-x-3=0$; and similarly $\frac{3 \pm \sqrt{-23}}{4}$ is nothing but $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ of the equation $2x^2-3x+4=0$.

It may be brought to the notice of the students that comparison between the three equations (two numerical and one general) is the basis in this regard. Let them write the equations side by side as under and observe :—

$$\begin{array}{ll} x^2-x-3=0 & a=+1,\ b=-1,\ c=-3 \\ 2x^2-3x+4=0 & a=2,\quad b=-3,\ c=4 \\ ax^2+bx+c=0 & \end{array}$$

The observation of a few more numerical quadratic equations and their roots, will convince the students about the application of the formula :—

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

It is, of course, to be added to the knowledge of the students that b^2-4ac is known as Discriminant. This 'discriminant' is an important quantity. It governs the nature of the roots. If it is zero, then $x=\frac{-b}{2a}$, that is, both the values of x will be $=\frac{-b}{2a}$ or the roots will be equal. If the quantity b^2-4ac is possible, as in the case of the first example, $\pm\sqrt{b^2-4ac}$ will be possible or real quantities and so the roots will be real. If the quantity b^2-4ac is negative, as in the case of the second example, $\pm\sqrt{b^2-4ac}$ will not be possible and so the roots will be imaginary.

Some practice in finding the nature of the roots can be given with the help of the following table, where the students will not actually find the roots but come to know the nature of the roots from the discriminant alone.

Equation	Discriminant b^2-4ac	Nature of Roots		
		Real or Imaginary	Equal or Unequal	Rational or Irrational
$x^2+6x+5=0$				
$3x^2+7x+2=0$				
$x^2+7x+6=0$				

Fig. 37.

EXERCISES

1. Enlist and explain the aims of teaching algebra in schools.
2. By referring to a few important topics of algebra, explain

the procedures which you would like to adopt for teaching this branch.

3. How will you teach algebra by correlating it with other branches of the subject of mathematics ?

4. Algebra is generally shunned by a student, why ? How will you create interest in its teaching ?

CHAPTER NINETEEN

THE TEACHING OF GEOMETRY

GEOMETRY

GEOMETRY IS THE science of space and extent. It deals with the position, shape and size of bodies but has nothing to do with their material or physical properties.

DEMONSTRATIVE GEOMETRY

It deals with the shape, size and position of figures by pure reasoning, based on definitions, self-evident truths, assumptions and other established geometrical truths. Euclid, a celebrated Greek Mathematician (323—285 B.C.), was the father of Demonstrative Geometry. He devised many methods for handling its problems. His methods are intuitional, observational, inventional, constructive, informal, creative, experimental and so on.

PRACTICAL GEOMETRY

It covers the construction work of the subject. Of course, most of this work is also directly or indirectly based on demonstrative geometry.

WHY TEACH GEOMETRY

1. It enables the learner to acquire a mass of geometrical facts.
2. The geometric principles of equality, symmetry, and similarity are implanted in the very nature of things.
3. It is important in a person's cultural development.
4. It develops the ability to draw accurate plans.
5. It provides a content that is objective and non-controversial.
6. It is useful in engineering, machine-shop, construction industries, landscape architecture, interior decoration and other areas of appreciation.

7. It demonstrates the nature and power of pure reason.
8. It is the key to mathematical thinking.

Some of its special functions at different stages of schooling can be further discussed.

ITS FUNCTION IN THE MIDDLE SCHOOL

The teachers of mathematics and curriculum-framers have now begun to realise that nature and practical arts are the primary and permanent sources of geometric learning. The function of the middle school geometry is to systematise the information received by the pupils at the pre-school and primary school stage from nature and practical arts. The emphasis will be on the understanding of fundamental concepts and techniques such as the meaning, drawing and use of lines, angles, triangles and polygons. The primary object is not to teach the pupils to know geometry; but rather to lead them to think geometry. The practical side of geometry, will, however, dominate at this stage.

ITS FUNCTION IN THE HIGH SCHOOL

The major purpose of geometric instruction at this stage will be to inculcate in the pupils an appreciation for logical demonstration; to acquaint them with effective methods of clear, impartial thinking, critical evaluation, and intelligent generalisation; to train them in the discovery of truth ; and to introduce them to the meaning of geometrical rigour and precision. From a study of demonstrative geometry, every pupil should acquire a comprehensive knowledge of geometric facts, concepts, and processes ; an intimate acquaintance with the nature of deductive reasoning ; and some facility in the application of geometry to the better interpretation and appreciation of one's environment.

THE FUNCTION OF GEOMETRY FOR THE SENIOR HIGH SCHOOL STUDENTS

For those who continue their study of geometry in the senior high school, intuition and experiment will still be effective aids. But the major purposes of instruction will be (a) to instil in the pupils an appreciation for the significance of logical demonstration; (b) to acquaint them with effective methods of clear, impartial thinking, critical evaluation, and intelligent generalisation ; (c) to train them in the techniques of discovery of truth; and (d) to introduce them to the significance of rigour and accuracy.

Geometry attains its maximum possibilities if, in addition to its direct and practical usefulness, it can establish a pattern of reasoning. The study of demonstrative geometry should enable every pupil to acquire a comprehensive knowledge of geometric facts, concepts and processes : an intimate acquaintance with an effective method in detached arguments and in the discovery of truth ; and some facility in the application of geometry to the better understanding of one's environments.

Therefore, increasing importance is to be given to geometrical instruments and their use. The values to be gained from a study of geometry are to be pointed out : formulae have to be derived and used ; the general and historical notes have to be given ; illustrations have to be used—all these have to be done as frequently as possible. To give emphasis to the need for training in logical thinking, more use has to be made of incomplete proof. There have to be more informal introductions through simpler language and less formal style. The subject-matter has to be presented through practical applications of geometric principles. There has to be less emphasis on strict logic than before, and the subject-matter has to be presented from the point of view of the yet immature pupil.

DIFFERENT STAGES OF TEACHING GEOMETRY

This logical subject is to be taught in a sequence. The following three stages are suitable and useful for its teaching:—

1. THE PRACTICAL STAGE

Here the pupil will acquaint himself with the common geometrical concepts and figures. Let him examine and handle geometric models, straight lines, curved lines, angles, triangles, polygons, circles, cubes, cuboids, cylinders, spheres etc. The pupil will be mainly guided to experience the symmetry, variety, regularity and beauty of forms in nature and practical arts. He will be taught how to keep and handle the instruments of his geometry box. The work will centre round the observing and drawing of common geometric figures. It is the stage of the geometry of the class-room and environment. It is the period of experimentation, observation, recognition and construction. But it should not be taken to mean that practical geometry ends with this stage. It is the beginning of the entire geometrical work.

2. THE STAGE OF REASONING

It is the stage to learn to prove theorems and exercises. The proofs will have to be presented in both the practical and theoretical forms to provide flawless understanding. In the latter part of this stage, the students will be expected to get used to reasoning without dependence on real and concrete instances. The reasoning will more and more incline to the side of argument than stick to the obvious nature of observation. Informal reasoning will be encouraged and made interesting and attractive at this stage. The pupil will be enabled to know the interesting theorems of plane geometry and to solve easy riders. He will be prepared for the more formal reasoning to come in the next stage.

3. THE SYSTEMATISING STAGE

It is the stage of acquisition of mastery in reasoning. The reasoning here will be morigorous but properly suited to the mental age of the pupils. Practice in logical reasoning will be more important than convincing them that the facts are true. Dependence on axioms will also be reduced.

WHEN TO INTRODUCE REASONING

Even in the beginning, the child should be allowed to use logic whenever he can. Actually the child begins to use reason at quite an early age. The children who can see the logical connection between facts should be encouraged to do so. But it is not to be forgotten that the main function at the early stage is to collect and experience facts. If the young child does not respond to logic satisfactorily, there need not be any hurry or worry about it.

SOME MORE TEACHING POINTS:—

1. The practical work should precede and clarify logical work.
2. The teacher should depend on first-hand experience and visual aids.
3. The practical work should be very neat and evident.
4. The pupils should be asked to observe things themselves by actual measurements or experiments. These should be no oral teaching of definitions and memorising of abstract ideas.
5. The blackboard work should be sufficiently neat, clear and accurate to avoid doubts and misconceptions. The teacher should also insist upon the accuracy of language. Coloured chalks may be used to emphasise significant details.
6. Analytical approach in the beginning should become a rule with the pupils. Synthetic approach should not be applied unless and until the idea is thoroughly understood.
7. A proper atmosphere should be created in the class-room by displaying suitable charts and models on the subject.

8. The subject should be frequently correlated with arithmetic and algebra.
9. The riders should not be left to be done at the end of the session. They should be done side by side with the theorems.
10. In oral work avoid the use of letters such as A, B, C for the pupil has to strain to decide which letter was said, consequently his attention must flag.
11. In the middle of an oral lesson it is a great relief to the class to have to write for a few minutes; they find it easier to attend after this brief change.
12. The boys should be allowed to use abbreviations from the start provided that they really understand them. But this use should be limited to a legitimate extent. For instance, abbreviations should not be permitted in the enunciation of a theorem.
13. The hand of the pupil should be trained to draw figures for theorems and riders freehand and with ease. But in the beginning, all constructions should be made with the ruler and compass.
14. Students should occasionally be asked to write from memory the enunciations of as many theorems as they can.
15. The pupils should write in their note-books the few definitions which they have to memorise.
16. It is very useful to get a class think of the changes that take place in a figure as some element in the figure is changed.
17. It is essential that geometrical vocabulary is used in correct sense. Technical phrases should be used only after understanding them.
18. There should be as little use of a text-book as possible.
19. Pupils should be encouraged to express themes by drawing, by construction and by words as fully as possible.
20. Let the pupils generalise the final results themselves.

AXIOMS AND POSTULATES

AXIOMS

An axiom is a mathematical fact to be accepted without proof.

The part is smaller than the whole; halves of equals are equal ; in an equation or an inequality, a quantity can be substituted by its equal ; things which can be made to fall exactly upon one another (things which can be made to coincide) are equal—such self-evident truths in geometry are called axioms. They require no proof.

POSTULATES

A postulate is a self-evident problem assumed without proof. A straight line may be produced (made longer) towards either of its extremities ; a circle may be drawn with any point as centre and any length as radius ; a straight line has one mid-point and only one; two straight lines cannot intersect in more than one point ; two intersecting straight lines cannot both be parallel to the same straight line.

Such axioms and postulates should be taught and understood through actual measurements and experimentation. They should be consistent, free from doubts, and should present no conflict with established knowledge or obvious facts. They are like the rules for the learner of geometry. They are accepted as universally true because of their conformity with common experience and sound judgment. The entire subject of geometry revolves round axioms, postulates and definitions ; hence these are frequently called the bases of geometry.

SOME FUNDAMENTAL CONCEPTS IN GEOMETRY

These constitute some basic ideas and terms of geometry. These concepts have to be made clear to the students. Their conception is to precede their definitions. Definitions should be given only after the concepts have been thoroughly understood.

1. SOLID

Anything that occupies space is called a solid. A solid has three demensions, viz., length, breadth and thickness.

The teacher should illustrate a solid with the examples of a brick, a book, a pillar, a stock, etc.

2. SURFACE

A boundary of solid is called a surface. A surface has length and breadth but no thickness. This idea can be illustrated with the examples of a table top, a floor, a human body, a swimming tank etc.

3. PLANE SURFACE

A surface is plane if it is perfectly flat, level and even ; such as, table top, floor etc.

4. CURVED SURFACE

If a surface is not perfectly flat and even, it is a curved surface ; such as, the surface of an uneven heap of wheat, the surface of a football, etc.

5. LINE

It is a boundary of a surface. It has only one dimension,—length. A line is that which has length, but neither breadth nor thickness.

The best method to explain it is to draw a line before the pupils.

6. STRAIGHT LINE

It is that which has the same direction from point to point throughout its whole length. It is the shortest distance between any two points. Only one straight line can be drawn between two points.

Place a foot rule and draw a line with it to show the shape and nature of a straight line. The name of a straight line can also be indicated by writing two letters, one at each end.

7. CURVED LINE

A line which does not have the same direction from point to point throughout its whole length is called a curved line.

This concept should be explained by drawing an arc or a line with bends.

8. POINT

It is the boundary or the extremity of a line. It has, therefore, positionbut no magnitude. A point is that which has position, but has no length, no breadth, no thickness. It is named by a single capital letter.

It should be shown by a dot, or by two intersecting straight lines. It is another name for the full stop.

9. ANGLE

An angle is the inclination of a straight line on another. It is formed by the rotation of a straight line about its end. The amount of rotation, or the angle, does not depend upon the length of the line (arm).

The idea of angle should be given with the help of a part of compasses, the hand of a toy watch, the opening of a door, a pair of scissors, and the opening of a book. Its actual diagram on a chart or blackboard will be a more effective demonstration. The angle in the diagram has two names : ∠ABC and ∠B. An angle consists of three elements : vertex or angular point B and two arms AB and BC. But where more than one angle is formed at the same point, each angle should always be denoted by three points.

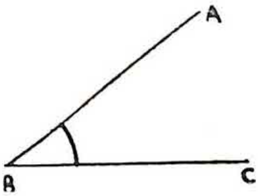


Fig. 38

10. ADJACENT ANGLES

Angles which lie on either side of a common arm are called adjacent angles. The opposite figure illustrates that ∠ABD and ∠DBC are adjacent angles. They lie on either side of the common arm BD.

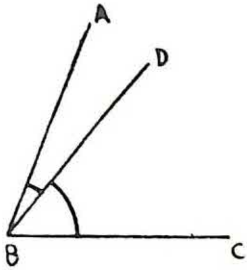


Fig. 39

11. STRAIGHT ANGLE

It is an angle whose arms extend in opposite directions from the vertex, forming a straight line. It is equal to 180° or two right angles.

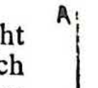


Fig. 40 (a)

12. RIGHT ANGLE

It is an angle whose one arm stands straight or vertical on the other. The straight line which makes a right angle with another straight line is called perpendicular to it—such as, AB is perpendicular to (⊥) BC. A right angle is equal to 90°.

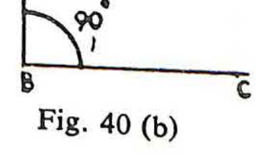


Fig. 40 (b)

13. ACUTE ANGLE

An angle which is less than a right angle is called an acute angle. Numerical questions can also help in bringing home to the students the idea of this angle. For example what is the name for an angle of 70°?

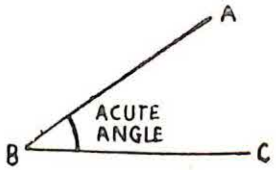


Fig. 41

14. OBTUSE ANGLE

An angle which is greater than one right angle but less than two right angles is called an obtuse angle. This angle lies between 90° and 180°. It should be recognised by the students that the angles of 91°, 100°, 105°, 130°, 150°, 170°, and 179° all belong to this category.

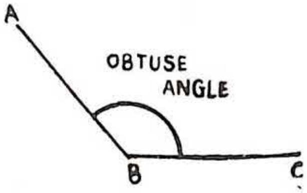


Fig. 42

15. REFLEX ANGLE

An angle which is greater than two right angles but less than four right angles is called a reflex angle. The idea should be illustrated by giving examples of angles of 270°, 300°, 330° etc.

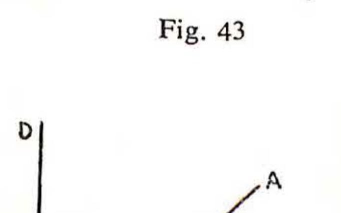


Fig. 43

16. COMPLEMENTARY ANGLES

Two angles are complementary when their sum is one right angle. The angles of 60° and 30° are complementary angles. In the diagram the angles ABC and ABD are complementary.

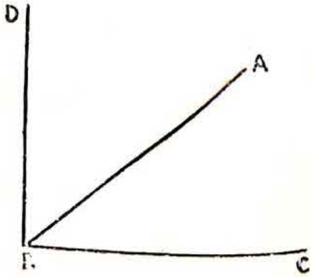


Fig. 44

17. SUPPLEMENTARY ANGLES

Two angles are supplementary when their sum is two right angles. The angles of 120° and 60° are supplementary angles. In the diagram, the angles ABC and ABD are supplementary. Numerical examples are the most suitable medim of practice. 'For example, Find the complement and supplement of 30°, 45°, 75° etc.'

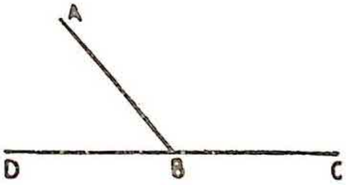


Fig. 45

18. ANGLES AT A POINT

The angle at a point is equal to four right angles or 360°. There may be any number of angles round a point, but their sum will be 360°. In the diagram, there are six angles at the point O.

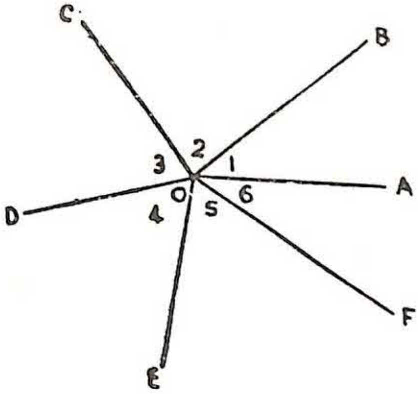


Fig. 46

19. VERTICALLY OPPOSITE ANGLES

Two non-adjacent angles which are formed by two intersecting straight lines are called vertically opposite angles. In the figure, ∠1 and ∠2 from one pair of vartically opposite angles, and ∠3 and ∠4 from another pair. It should be shown by actual construction and measurement, that these angles are equal.

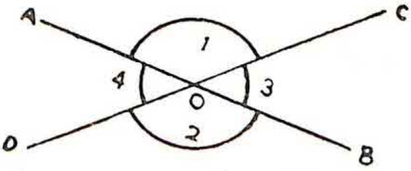


Fig. 47

20. ALTERNATE ANGLES

When a third straight line (transversal) intersects two parallel straight lines, as shown in the figure, the angles 1, 3 and 2, 4 are called alternate angles.

Let the students find out by actual construction and measurement that alternate

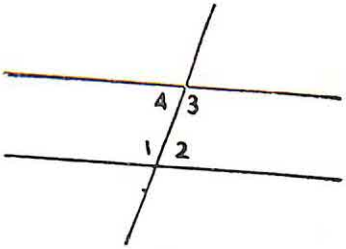


Fig. 48

angles are equal. $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$.

Exercise : If $\angle 1 = 110°$, find out the values of the remaining three angles ?

21. CORRESPONDING ANGLES

When two parallel straight lines are intersected by a third straight line (transversal) the pairs of corresponding angles thus formed are (1, 5), (2, 6), (3, 7) and (4, 8). Actual measurements can show that the corresponding angles are also equal.

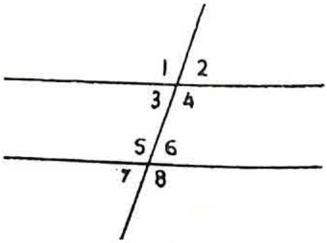


Fig. 49

Exercise : If $\angle 1 = 120°$, find the remaining seven angles ?

Exercise : The teacher will draw two pairs of parallel lines and then write pairs of angles as shown in the figure. The students will be required to recognise the pairs of alternate and corresponding angles.

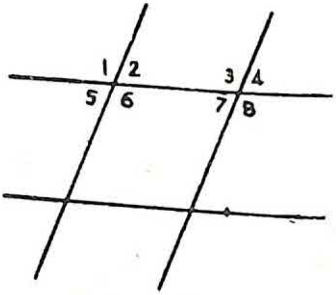


Fig. 50

22. PARALLEL STRAIGHT LINES

Straight lines which are in the same plane and do not meet however far they may be produced both ways (towards both directions) are called parallel straight lines. The two rails of the railway line are parallel in the real sense.

23. POLYGON

A polygon is a rectilinear figure. It is a closed broken line. The figure opposite shows a polygon with sides AB, BC, CD, DE and EA, and with angles A, B, C, D and E.

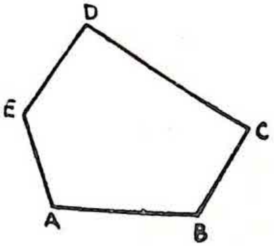


Fig. 51

24. REGULAR POLYGON

A polygon which has all sides and all angles equal is called a regular polygon. The polygon to be commonly used are pentagon (5 sides), hexagon (6 sides), heptagon (7 sides) and octagon (8 sides).

25. PERIMETER

The sum of the sides of a rectilinear figure is called its perimeter. Actual measuring opportunities should be used to illustrate this fact. The figure opposite shows the perimeter $=3''+3''+4''+5''+2''+3''+''=20$ inches.

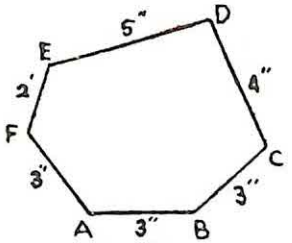


Fig. 52

26. TRIANGLE

It is a polygon having three sides. Every triangle has six parts, three sides and three angles. Any one of the sides may be taken as its base. The base of a triangle is the side on which it is supposed to stand. The two angles B and C are called base angles and $\angle A$ is called the vertical angle. The pt. A is the vertex. The triangle being the most important of the geometric figures studied in plane geometry, should be thoroughly illustrated, constructed or grasped.

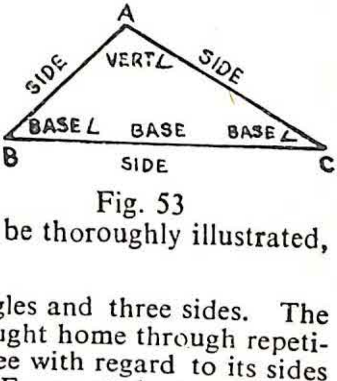


Fig. 53

A triangle has six elements—three angles and three sides. The significance of each element should be brought home through repetition and practice. It has six kinds (i) three with regard to its sides and (ii) three with regard to its angles. Every student should be required to prepare a chart on all kinds of triangles.

27. EQUILATERAL TRIANGLE

A triangle which has all of its sides equal is called an equilateral triangle. In $\triangle ABC$, $AB=BC=CA$.

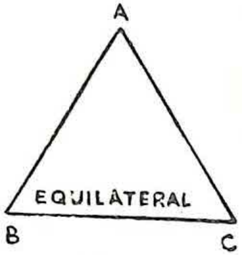


Fig. 54

28. ISOSCELES TRIANGLE

A triangle which has two of its sides equal is called an isosceles triangle. In △ABC, AB=AC.

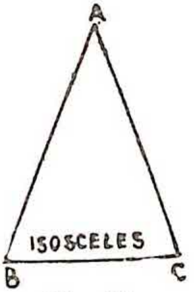


Fig. 55

29. SCALENE TRIANGLE

A triangle with no two sides equal is called a scalene triangle. For example a triangle with sides 2.5″, 3″, 3.5″ is a scalene triangle.

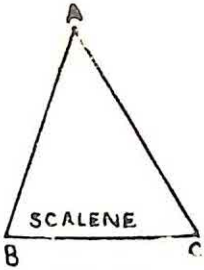


Fig. 56

30. A RIGHT-ANGLED TRIANGLE

A triangle which has one right angle is called a right-angled triangle. The △ABC is right-angled at B.

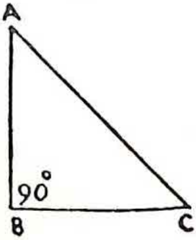


Fig. 57

31. OBTUSE-ANGLED TRIANGLE

A triangle which has one obtuse angle is called an obtuse-angled triangle. The △ABC is obtuse-angled at B.

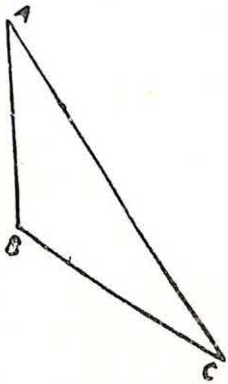


Fig. 58

32. ACUTE-ANGLED TRIANGLE

A triangle which has all of its angles acute is called an acute-angled triangle. A triangle with angles of 65°, 70° and 45° is an illustration.

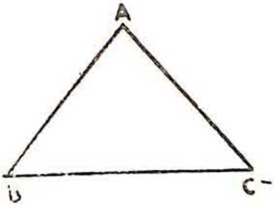


Fig. 59

33. MEDIAN

The straight line joining the vertex of a triangle to the mid pt of the base is called 'median.' In $\triangle ABC$, AD is the median. There can be three medians in a triangle.

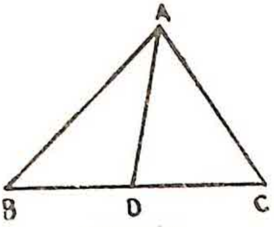


Fig. 60

34. CONGRUENT TRIANGLES

Any two geometric figures are said to be congruent if they can be made to coincide (fit exactly on each other). Congruent triangles are those which are equal in all respects.

35. EQUIANGULAR TRIANGLES

When the three angles of the one are respectively equal to the three angles of the other, the triangles are equiangular. They are also known as similar triangles.

36. QUADRILATERAL

A polygon with four sides is called a quadrilateral. The figure ABCD is an illustration.

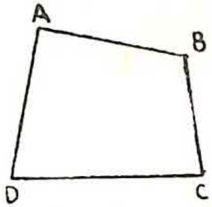


Fig. 61

37. PARALLELOGRAM

It is a quadrilateral which has pairs of opposite sides parallel. In figure ABCD, AB ∥ CD and AD ∥ BC.

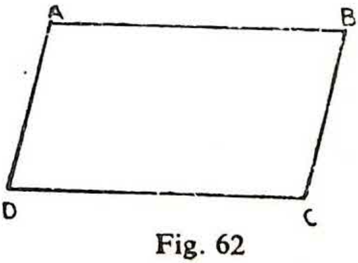


Fig. 62

38. RECTANGLE

It is a parallelogram having one right angle (hence all ∠s are rt. ∠s.)

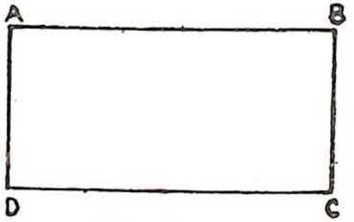


Fig 63

39. SQUARE

It is a rectangle which has all sides equal. In fig. ABCD, $\angle A=90^\circ$ and $AB=BC=CD=DA$.

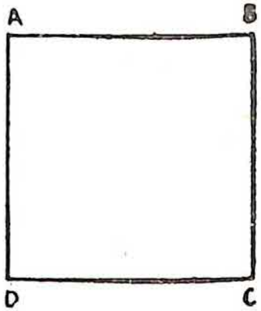


Fig. 64

40. RHOMBUS

It is a parallelogram which has all sides equal, but its angles are not right angles. In fig. ABCD, $AB=BC=CD=DA$.

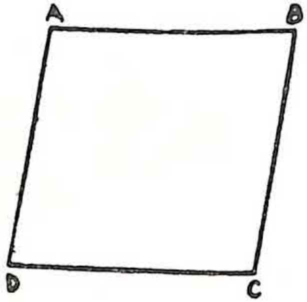


Fig. 65

41. TRAPEZIUM

It is a quadrilateral having one and only one pair of parallel sides. In fig. ABCD, $AB \parallel DC$.

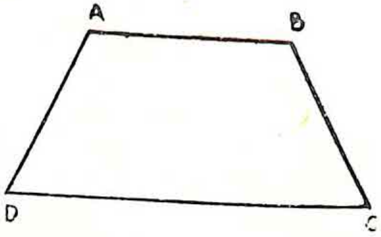


Fig. 66

42. DIAGONAL

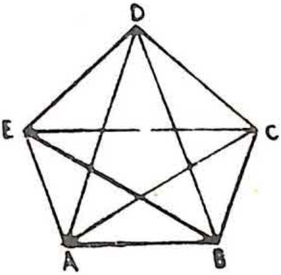


Fig. 67

A straight line joining the opposite verticals of a figure is called its diagonal. In the figure ABCDE, AC, BD, CE, AD and BE are all diagonals.

43. CIRCLE

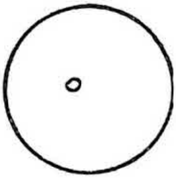


Fig. 68 (a)

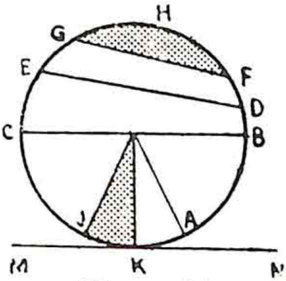


Fig. 68 (b)

It is a closed plane curve, all points of which are equidistant from a point within, called the centre. A circle has a large vocabulary of its own. A beautiful coloured chart of the 'vocabulary of circle' should decorate the walls of all mathematics rooms. Fig. 68 (a) shows a circle with centre O and Fig. 68 (b) shows various terms connected with a circle.

44. CIRCUMFERENCE

The line by which a circle is bounded is called the circumference of the circle.

45. CENTRE

It is the fixed point from which every point on the circumference is equally distant. In the above figure, O is the centre.

46. RADIUS

The straight line which joins any point on the circumference to the centre is called the radius of the circle. Here OA is the radius.

47. ARC

Any part of the circumference of a circle is called an arc of the circle. CE is the arc in the figure.

48. CHORD

A straight line which joins any two points on the circumference of a circle is called a chord of the circle. DE is a chord in the figure.

49. DIAMETER

A chord of a circle passing through the centre of a circle is called the diameter. Here BC is the diameter.

50. SEGMENT OF A CIRCLE

It is the part of a circle bounded by a chord and the arc which it cuts off. In the figure, FGH is the segment.

51. SECTOR

A part of a circle bounded by two radii and the arc intercepted by them is called a sector of the circle. In the figure OJK is the sector.

52. TANGENT

It is the straight line which touches the circumference of the circle at one and only one point. In the figure, MN is the tangent.

53. CONCENTRIC CIRCLES

The circles with the same centre but with different radii arc called concentric circles. The figure below (left-hand) shows concentive circles.

54. INTERSECTING CIRCLES

The circles which intersect are called intersecticg circles. The figure opposite (right-hand) shows intersecting circles.

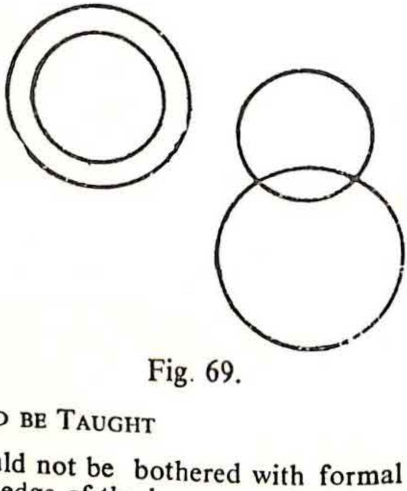


Fig. 69.

WHEN AND HOW DEFINITIONS SHOULD BE TAUGHT

In the early stages a learner should not be bothered with formal definitions; he needs a working knowledge of the language of geometry; he must understand the meaning of terms and expressions rather than be able to define them in a technical language. In many cases, actual drawings of these terms and expressions are easier to understand than their word pictures or definitions. A pupil at this stage should be concerned with a clear conception of what the thing is—a formal definition does not help him much. Technical definitions or phrases are sometimes too philosophical to be appreciated by the learner.

When a new term occurs in geometry, a grasp of its pictorial form should be given first. Its pictorial meaning and qualities should be translated into the necessary definition or language. In certain cases, the definition should be introduced after the term has been used and its drawing has been made a number of times.

Definitions can be divided into several classes :

(i) *Philosophical Definitions*, for example, of point, straight line. These are quite difficult to understand, but when they are put in their fundamental forms the idea becomes clear. To introduce the idea of point or straight line with the definition is an unpsychological beginning.

(ii) *Explanatory Definitions*, for example, of vertical angles, corresponding angles, chord of a circle. Let the child observe their shapes first, then he may construct them and finally learn how to define them.

(iii) *Definitions, which are essential for logical purposes*, e.g., of equilateral triangle, rectangle, rhombus. Their shapes and measurements will be a good introduction.

These definitions, after they have been observed and understood, should be learnt by heart. There may be more than one definition or explanation of a term; but for a logical system it is essential to agree on one standard set as fundamental.

The teacher should always remember that the definition is the explanation of a term by means of others which are more easily understood. So the explanation should not be more difficult and confusing than the term being defined. It should rather be simpler and more easily understandable. Its essentials are simplicity, clarity and brevity. It must be as short as possible and be free from ambiguity. It should dispel the existing doubts rather than create any new doubts.

The mastery of definitions is always a tiresome and most uninteresting work for the pupils. If the start is made with definitions, it shall be giving the pupils memory work without properly illustrating the significance of the term. Too much insistence on rigour in memorising definitions is not going to promote the original thinking of the pupils. The learning will be fruitful only when the pupils themselves arrive at definitions. They should be based on concrete experiences and their treatment in problems. There is no need of burdening the pupils' brains with the definitions of the terms which are acceptable and obvious without definitions. Some terms are not easily definable; any attempt to define them will not be profitable, and may result in confusion, e.g., locus, circle etc. Emphasis should always be laid on understanding rather than on memorised statements. Formal definition should form the end and not the starting point of the study of the term.

HOW TO TEACH PROPOSITIONS

The propositions in geometry can be divided into two classes : (i) Theorems and (ii) Problems.

In a theorem, a statement is to be proved, while in a problem some geometric construction is to be made or some computation to be done.

TWO PARTS OF THE THEOREM

Every theorem is generally expressed as a complex statement having two parts : (i) the 'if' part, beginning with "if". It is the given part or the known part of the theorem. It is also called the Hypothesis, and (ii) the 'then' part, beginning with "then". It is the part 'to be proved'. It is the unknown part of the theorem and is called the 'Conclusion'. The 'if clause' contains the condition which is given or assumed to be true for the sake of argument, and the 'then clause' states what is to be proved.

It is customary and desirable for the learner to divide his work on a proposition into the following stages, which constitute the formal proof :—

(i) Statement of the theorem.
(ii) The figure.
(iii) A statement of what is given (Hypothesis).
(iv) A statement of what is to be proved (Conclusion).
(v) Construction, if any.
(vi) Orderly proof.

THE STATEMENT OF THE THEOREM

The statement is written as it is given in the problem. The student must understand the enunciation clearly.

The Figure is to be constructed by noticing the purpose of the theorem. It should be seen whether the theorem concerns intersecting st lines, parallel st. lines, two triangles, a rt. ∠d triangle or a circle. The learner should be encouraged to construct the figure on his own. The figure should be drawn in a different position and lettered differently from the way given in the book.

The Hypothesis can be easily located from the statement. It has to be written in terms of the st. lines, points, angles etc., of the figure constructed in the previous step. There may be many obvious elements in this figure, but only those are to be written which are quite essential conditions of the theorem. The 'given' or 'known part' of the statement should be understood by the pupil before it is written. It should be written in a brief manner as far as possible, such as, instead of writing 'Two parallel lines AB and CD are intersected by the transversal EF respectively in G and H,' the student can write the given as AB ∥ CD, transversal EF intersecting them in G and H.

The Step of 'to Prove' or 'Conclusion' is also to be written by the pupil in his own words. This should be recorded in very clear terms.

The Proof is the final or major part of the work. It consists of a number of reasons and arguments. The student is to proceed step by step, by giving sufficient reasons for each step. The arguments have to be based on the items considered as "given", definitions, axioms, postulates, propositions already proved, identities or construction. The proofs have to be made as simple and short as possible.

A theorem can be proved in many different ways. Hence, the teacher should not insist on having from the student the proof given in the text-book. The pupils should be encouraged to adopt their own ways of proof. It is not necessary that the arguments should be in the same order as given in the book.

The analysis of the statement should be the guiding factor in writing all the above steps. Premium should be placed on intelligent and thoughtful reasoning rather than on cramming.

When the pupil has understood a proposition, he should be asked to reproduce it independently. The student should be asked to form the habit of running over the proposition mentally, making a mental figure.

The theorem must be applied to exercises and used in solving them. As many applications and exercises as can be done conveniently should be given.

KINDS OF PROOFS

It is interesting and psychological for the child to be allowed to follow different kinds of proofs in different cases. Sometimes one kind of proof is more agreeable than the other, therefore, insistence on any one type exclusively is not a healthy approach.

THE EXPERIMENTAL PROOF

(i) Experimental proof is, naturally, very convincing and satisfying. The equality of the vertically opposite angles can be easily shown by actually measuring them. Similarly the angle subtended by an arc at the centre can be measured and shown as double of the angle subtended by the same arc at the circumference. In this case, theoretical argument is replaced by practical processes. It will be in the interest of the child, if the experimental proof is taught before the logical proof of any proposition. It is the only method in certain propositions such as "the congruency of triangles".

THE LOGICAL PROOF

(ii) It is a process of pure, abstract argument in most cases. It is given systematically in four distinct steps : (i) Data, (ii) What is to be proved, (iii) Construction, and (iv) Proof to develop efficiency and speed in 'logical proof' the pupil has to bear in mind the knowledge of geometrical concepts, axioms, postulates and the propositions already proved This proof proceeds from 'unknown' to 'known'. It has to be developed with the help of well-framed short questions.

THE INTUITIVE PROOF

(iii) It is not applicable to all theorems. In some cases, the proof is almost obvious, there is no need of experimental or logical proof. For instance, the propositions : 'The sum of any two sides of a triangle is always greater than the third' or 'Parallel straight lines never meet' fall in this category. Of course figures, observation of figures and measurements can be convincing processes in such cases. The proof may look obvious, but it has ultimately to be written.

INDIRECT PROOF

(iv) This method is applicable to only a few theorems. It is difficult for many pupils to understand. It is used to prove those propositions which cannot be easily proved by the direct method. The propositions, such as—'If a straight line cuts two other straight lines, making a pair of alternate angles equal, the two straight lines are parallel'—fall in this category. Here the student has to start with the assumption, 'if the two lines are not parallel, let them meet at a point, when produced to the left or right.' This alternative or assumption is shown to be impossible, and hence the theorem stands proved indirectly.

REDUCTIO AD ABSURDUM METHOD

(v) It is also a form of the indirect proof. The student starts with the stated conclusion assuming it to be untrue for the sake of reasoning; reads an impossibility or an absurdity and consequently infers that the assumption with which he started was wrong and untrue. When the assumption is untrue, the opposite is accepted as true. This method is quite useful in proving the converse theorems. For example : 'If two adjacent angles are supplementary, their outer arms are in the same straight line.' The student starts with the assumption that the two arms are not in a straight line; he rather constructs an alternative st. line by producing one of the arms. The absurdity of his construction and assumption soon becomes obvious to him. Hence it cannot but be accepted that the arms are in a straight line.

PROOF OF EXHAUSTION

(vi) It is also another form of indirect proof. Here he takes all the possible cases, and then shows that each of these suppositions is false, with only one exception. Thus the exceptional case is accepted as a true case. For example ; 'If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.' The teacher says that if the particular side is not greater than the other side, it must either be less than or equal to it. Both the latter possibilities are exhausted by showing them contrary to something known. Only one possibility is left, and that is consequently accepted.

CONVERSE, CONTRA-POSITIVE, RECIPROCAL AND INVERSE THEOREMS

The learner should be made familiar with these to enable him to understand the application of indirect method for proving geometrical truths by analysis. He will then understand the interdependence and inter-relationship of theorems, and always look for the possible help he can obtain from already proved propositions for proving a new one.

THE CONVERSE THEOREM

(i) Two theorems are said to be converse of each other, when the 'known' of one becomes the 'unknown' of the other and vice-versa. The hypothesis and the conclusion change their places.

For Example

THEOREM

In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the two sides.

ITS CONVERSE

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the triangle is right-angled.

Here the converse is also true, but in some cases the converse may not be true, e.g.—

THEOREM

If a triangle is equilateral then it is isosceles.

ITS CONVERSE

If a triangle is isosceles, then it is equilateral.

It is, therefore, obvious that the truth of the converse of every theorem cannot be taken for granted. It is to be accepted only after a separate proof.

THE CONTRA-POSITIVE THEOREM

(ii) In this the positive nature of both the hypothesis and conclusions is changed to negative. The contradiction of the conclusion is taken as the new hypothesis, and the contradiction of the hypothesis is to be the new conclusion.

For Example

THEOREM

In a circle, if two chords are equal, they cut off equal arcs.

ITS CONTRA-POSITIVE

In a circle, if two chords cut off unequal arcs, the chords are also unequal.

The contra-positive is obviously true.

THE RECIPROCAL THEOREM

(iii) If in a theorem on a triangle, a point is changed into a line, and a line into a point, angles by the opposite sides, and the sides by the opposite angles, then it becomes the reciprocal of the original.

For Example

THEOREM

If two sides and the included angle of one triangle are equal to two sides and the included angle of the other, the two triangles are congruent.

ITS RECIPROCAL

If two angles and the included side of one triangle are equal to two angles and the included side of the other the two triangles are congruent.

The reciprocal, in this case is quite true; but it does not apply to all theorems.

THE INVERSE THEOREM

(iv) It is also known as Opposite Theorem. It is obtained by taking the contradiction of the hypothesis as the new hypothesis and the contradiction of the conclusion as the new conclusion.

For Example

THEOREM

In a circle, if two chords are equal, they cut off equal arcs.

ITS INVERSE THEOREM

In a circle, if two chords are not equal, they cut off unequal arcs.

THE KEY PROPOSITIONS

The propositions have to be taught in a logical order. The order given in the text-book may not be a perfectly logical one. The teacher has to devise his own arrangement. The propositions have not to be treated as isolated and disconnected units. It is better, if they are taught in groups of allied propositions which show a marked degree of association, similarity and closeness. Similar ones should be grouped together, and their closeness and similarity should be adequately stressed.

In a group formed after careful sorting, there should be one key or parent proposition, serving as nucleus or basis of the other propositions of that group. The fundamental or basic nature of this parent proposition should be properly brought home to the pupils. They should know that it is to be used as a key to open the truths of more than one theorem.

The beginning will naturally have to be made with the key proposition. Other propositions will follow in their proper and logical order.

For Example :

(i) In the first few propositions on angles, the key proposition can be—

'If a st. line stands on another st. line, the sum of the adjacent angles so formed is equal to two right angles.'

(ii) In another group of a few propositions on the angles and sides of a triangle, the key proposition can be—

'If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.'

(iii) In the 'Tangent Group', the key proposition can be—

'The tangent at any point of a circle and the radius through the point are perpendicular to one another.'

(vi) In the 'Circle Group' the key proposition should be—

'Angle at the centre of a circle is double the angle at the circumference subtended by the same arc.'

Similarly, other key propositions may be found out. The work does not end here. Their link with other propositions of the group should be stressed as many times as possible. Thorough understanding of the key proposition will be of special help in learning other propositions of the group.

SYMBOLIC NOTATION

A good system of notation is at the disposal of teachers as well as students. The use of these symbols has made it possible to eliminate a great deal of verbal confusion and to shorten both written and oral exposition. Symbolic Notation saves labour, simplifies thinking and aids in comprehension and retention. It should be introduced from the very beginning. It makes expression orderly and to the point. It has become a convention to use capital letters to represent points and small letters to denote lengths ; the sides by AB, BC etc., and angles by ∠A, ∠B or ∠BAC, ∠ABC and so on.

FIGURE

In demonstrative geometry, the first thing is to translate the words of the statement into a figure and to mark in that figure all the known and the wanted data. The student should always draw his own figure while proving any theorem. The figure should be fairly large in size and reasonably neat. Though accuracy of the figure is not the main consideration here, yet straight lines should look straight, right angles right, parallel lines parallel and a circle circular. The given and the required should be marked in the figure which should be self-explanatory. Erasing should be avoided as far as possible because that will make the figure untidy. Even a freehand construction should be nearly accurate.

SIMPLE DIRECTIONS FOR LEARNING THEOREMS

1. (*a*) Read the problem carefully. Be sure that you understand its meaning.

   (*b*) Determine the hypothesis and the conclusion.

2. (*a*) Draw a fairly accurate and neat figure which should illustrate each point, angle and line described in the 'given'. (*b*) Letter the figure, marking the parts which are given equal.

3. Write what is given, in terms of the figure.

4. From the conclusion write what is to be proved, in terms of the elements of the figure.

5. To find out a method of proof for the theorem analyse by proceeding from the unknown to the known.

6. Record the proof, giving a reason for each step.

7. In revising or memorising a proposition, adopt the following course : (1) read the whole of the proposition, as given in the book, (2) repeat it with a figure drawn in a different position and differently lettered, (3) run over it mentally without looking at the figure.

8. In reproducing the proposition, you need not stick to the exact words of the book, nor be very particular about avoiding them. The figure need not be the same so long as it represents the data. The arguments need not be in the same order so long as the reasoning is sound.

REFERENCE

When giving reference of another proposition in the proof of a proposition, mere number of the proposition referred to does not suffice. A brief enunciation of the relevant proposition should be given. For example. In place of — $\angle 1 = \angle 2$ (vide Prop. 3) say— $\angle 1 = \angle 2$ (vert. opp. $\angle$s).

Difference between Teaching Euclid and Euclid Teaching: *Euclid Teaching* means the process of beginning with definitions, going through axioms, postulates and then arriving at the propositions. It is a deductive sort of process which was formulated by Euclid many centuries ago. He perhaps wrote his books for mature minds and not for young children. This approach suffers from the following drawbacks :—

(i) The order of learning is not psychological. Definitions etc., should come in the end, the stage when they can be formulated by the students.

(ii) The practical applications of geometry are neglected in this process.

(iii) Some of Euclid's ideas are crude ; for example, he assumed that angles must be less than two right angles.

(iv) It does not help the pupil in appreciating the basic characteristics of the perfect science of geometry.

Teaching Euclid: The term 'teaching Euclid' is a pedagogical invention. It means "Teaching what Euclid propounded". By this is meant the process of making the pupils understand the spirit behind Euclid's *Elements*. Euclid Teaching is the process of following Euclid blindly, whereas Teaching Euclid means teaching the child the geometry devised by Euclid. Teaching Euclid is a method for the young learner, whereas Euclid Teaching may be the scholar's domain. Therefore in schools Euclid should be taught. The teacher should not stick to Euclid Teaching.

INDUCTION AND DEDUCTION

The technique of bringing about the "transition from particular facts to a general knowledge about these facts is known as the process of induction." When the pupil measures vertically opposite angles in several cases and finds that in each case they are equal, he has the background necessary for the induction that "vertically opposite angles are equal". Such inductions are important in the discovery of truths whether geometrical in nature or not. But discoveries by induction are in the realm of probable truths. The students can only say, "it is probably true that vertical angles are equal".

Such experiments as these, of course, lay the foundations for inductive generalisations which in turn call for deductive demonstrations to establish their soundness and universality of application.

Deduction is "the process of following the network of relations which bind truths together." It is thus the process of drawing logical inferences from established facts or fundamental assumptions. Demonstrative Geometry is mainly a deductive science in which truths, stated in the form of theorems, can be proved by showing that they are implied by other theorems which have already been proved, definitions that have been stated, and postulates that have been accepted. In any deductive science, there will necessarily be a basic list of fundamental axioms and assumptions. These axioms and assumptions are agreed upon and accepted, because they are consistent, free from ambiguity and contradictions, and present no conflict with established knowledge or observable facts.

It should be evident to the thoughtful student that the intelligent study of geometry is both inductive and deductive in nature. The teacher should strive to keep before the student the challenge or inductive discovery and the assurance of deductive proof.

SYNTHESIS AND ANALYSIS

Benjamin Pierce defined mathematics as "the science which draws necessary conclusions". In the statement of this definition there is the essence of synthetic argumentation. Every geometric theorem has two characteristic properties, a hypothesis and a conclusion, or a known and an unknown. The proof of the theorem consists in the establishment of the truth of the conclusion through implications and inferences that find their original source of justification in the hypothesis. The synthetic proof consists of the drawing of a series of intermediary conclusions until the final and desired conclusion is reached. Although the simplicity, elegance, and rigour of this form of argument make it highly desirable, yet it is far from being desirable as a sole procedure to be followed in deriving geometric proofs. As a technique, it does not provide for the understanding of the reason for making necessary constructions or for applying different theorems. As a simple illustration, consider a synthetic proof for the theorem :—

If two chords of a circle interset internally, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

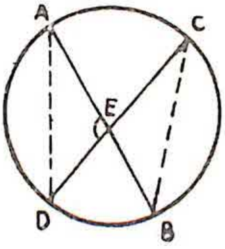


Fig. 70.

Given :— AB and CD two chords of a circle intersecting within the circle at E.

To prove :— $AE.EB = CE.ED$.

Const. : —Join AD and CB.

Proof :— $\because$ $\angle DAB$ and $\angle BCD$ are in the same segment of the circle.

$\therefore$ $\angle DAB = \angle BCD$.

In $\triangle$s AED and BEC,

$\because$ $\angle AED = \angle BEC$ (vert. opp. $\angle$s)

$\angle DAE = \angle BCE$ (proved)

$\therefore$ $\triangle$s AED and BEC are equiangular.

$$\therefore \frac{AE}{EC} = \frac{ED}{EB}$$

$\therefore$ By cross-multiplication

$AE.EB = EC.ED$.

Q.E.D.

The hypothesis, through the construction, implies the equality of the angles in the same segment, which, supported by the equality of two more angles of the same triangle (known on the basis of a previous proposition) leads one to conclude that the two triangles are equiangular. This conclusion further leads to the final conclusion. But why join AD and BC ? How did the author know that these lines should be drawn. Many questions such as these are likely to arise, and they are certain indicators of a lack of provision for understanding, clarity and self-confidence.

The analytic approach to a geometric proof consists in the search for sufficient conditions. The investigator looks at the conclusion of the theorem, and raises the question : What relation is sufficient to justify the use of this conclusion as a true statement ? Once this relation has been found he analyses it for the same purpose in the hope that he will ultimately arrive at the hypothesis of the theorem as the source of the chain of sufficient reasons. The steps of this process must be reversible.

The analytic approach to theorem given above would be as follows :—

1.	By what methods can be proved AE.EB=CE.ED ?	By possibly converting the equality of product into an equality of ratios, such as $\frac{AE}{CE}=\frac{ED}{EB}$.
2.	How can the equality of the ratios of these lines be shown ?	By taking some triangles containing them and showing the triangles similar.
3.	Which are the required triangles ?	Not available in Diagram. Therefore they have to be constructed.
4.	Which are the possible constructions ?	By joining AD and BC. [∴ Const. Join AD and BC]
5.	Now which are the two triangles ?	△s AED and BEC.
6.	Which are the obviously equal angles in them ? (Since the triangles have to be proved equiangular).	∠AED=∠BEC. being vertically opposite angles)
7.	Which is the other possible pair of equal angles ?	∠DAE=∠BCE [Since both the angles are in the same segment of the circle]
8.	What is the conclusion now ?	△s AED and BEC are equiangular.
9.	Where does it lead to ?	Therefore the corresponding sides are proportional.
10.	Write down the equations showing this proportionality ?	$\frac{AE}{EC}=\frac{ED}{EB}$. ∴ by cross-multiplication AE EB=EC.ED. Q.E.D.

Of course, the chain of implications can become much more complex than that indicated here ; but the basic pattern remains the same, regardless of the degree of complexity.

As already emphasised, the analytic process does not constitute a proof until the steps are shown to be retraceable in the reverse order, which constitutes the synthetic arrangement of the argument. Analysis helps to bring out the why of each step and synthesis helps to establish the elegance and rigour of the proof.

ANGLE-SUM OF A TRIANGLE

The class will be initially guided to reach this conclusion through experimentation by actually measuring the angles in a number of triangles and finding their sum in each case. Measurements and experiments in a limited number of cases will suggest a possible general truth. But it should be remembered that measurements in certain cases might not turn out to be in agreement with the supposed truth on account of some unavoidable but very insignificant inaccuracies in construction. The experimenter then should try to build up a theoretical proof.

Here are two interesting experiments :—

(1) Cut out a large paper triangle, tear off its corners, fit them together to make the angle-sum, and ultimately measure the angle-sum.

(2) Cut out a large paper triangle, fold its corners as shown in the figure to bring the three angles at a point so as to form one straight angle.

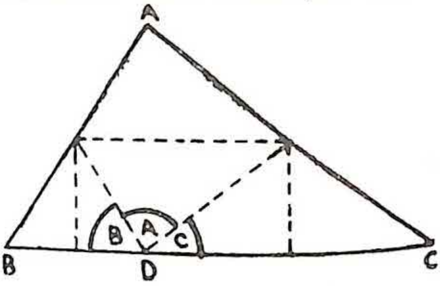


Fig. 71

After the practical work described above, the class can be led to the ordinary proof. The necessity of producing BC to D should be justified by explaining that the straight angle is constructed so that it may be shown equal to the sum of the three angles of the triangle. We construct a straight angle to equalise the three angles of the triangle with it. To begin with the equalising process the teacher can enable the students to observe that ∠C is already a part of straight angle BCD. The necessity of drawing CE ∥ AB can be shown at this stage that by this the remaining angle ACD is divided into two parts ∠ACE and ∠ECD which are respectively equal to ∠A and ∠B.

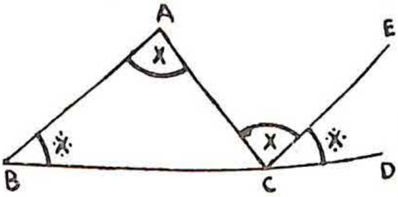


Fig. 73

The proof by drawing a parallel to the base right through the vertex should also be considered; but the proof given above is to be preferred, because it also proves, "The exterior angle formed by producing a side is equal to the sum of the two interior opposite angles."

At any time after this the class may be introduced to the various names applied to triangles : right-angled, acute-angled, isosceles, etc.

Naturally, the class can be led to the angle-sum of a quadrilateral by drawing any of its diagonals and dividing it into two triangles. This very method can be extended for finding the sum of the interior and exterior angles of any polygon. By drawing suitable diagonals, every polygon can be divided into a number of triangles. The angle-sum in every triangle being two right angles, this multiplied by the number of triangles formed gives the required result.

CONGRUENT TRIANGLES

Any two geometric figures are said to be congruent if they can be made to coincide. By this time the learner must have seen that the triangle is fixed if we are given :

(i) two sides and the included angle.
(ii) two angles and one side.
(iii) three sides.
(iv) a right angle, the hypotenuse and one other side.

The class will have actually constructed triangles from sufficient data supplied; but, at this stage, it is not a matter of accurate drawing. A conscientious teacher may provide each boy with a piece of tracing paper, make him draw a triangle with AB=3.5 in. AC=4 cm. ∠A=45°, and finally fit his triangle on to his neighbour's. He asks, 'Does it fit or not ?' and goes on to remark: "It must fit, if both the triangles are accurately drawn and are properly placed one over the other."

The proof of the congruency in this case can be best demonstrated through actual placement. Equal elements have to be placed and fitted on their equals in the second triangle. All the different cases can be proved by superposition. So it follows that congruent figures have the same size and shape. To the third and fourth cases, methods other than superposition are also applicable. They should also be introduced in the class.

SIMILARITY

It is a very important principle of elementary mathematics; but it has not yet achieved its rightful position. It has great mathematical simplicity and beauty about it, and has an extraordinarily wide range of application. Similar figures are related to one another in the same way as an object and its image or as a lantern slide and its projection on a screen or two maps of the same country drawn on different scales. These are figures of the same appearance, irrespective of their size.

There are two necessary conditions for similarity, namely, (i) The same shape (equal corresponding angles) (ii) Proportional dimensions.

These two conditions are adequate for the purpose.

'If two triangles are equiangular, their corresponding sides are proportional', can be treated as the 'key proposition' of the group of propositions on similarity. The proof of this proposition also is developed by the placement or superposition of the smaller triangle on the larger one, so that in the new position one of its sides is || to one of the sides of the larger triangle. This leads to the establishment of proportionality of the remaining two sides with the corresponding sides of the other triangle.

Similar polygons are polygons which have their corresponding angles equal and their corresponding sides proportional. The polygons P and P' are similar if $\angle A=\angle A'$, $\angle B=B'$ $\angle C=\angle C'$, $\angle D=\angle D'$, $\angle E=\angle E'$ and $F\angle=\angle F'$.

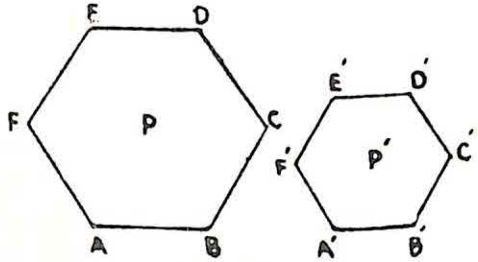


Fig. 73.

and $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\frac{DE}{D'E'}=\frac{EF}{E{\cdot}F'}=\frac{FA}{F'A'}$.

The principle of similarity helps in measuring the heights of poles, buildings, width of rivers and in enlarging and reducing maps.

LOCUS

Definition :—

The locus of a point moving under a given condition is the path traced out by it. From the definition it follows, "For the path of a point to be the locus the point must be moving under a given condition. If a point moves but no condition is attached to its motion, the path of the point will not be called locus."

The idea and significance of locus is not as widely understood by the students as it ought to be. They have been seen writing, "The locus of a point equidistant from two given points is on the straight line." The superfluous use of 'on' shows the lack of clarity of conception.

Even at the first reading locus should be studied carefully. The teacher should ask questions such as :—

(i) What is the locus of a man's hand as he winds the starting handle of a motor car ?

(ii) Where must the centre of a wheel lie, while the wheel moves on a straight track ? If the locus is the path or line in which a point must lie, what is the locus of the centre of the wheel ?

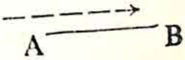
(iii) What is the locus of the pebble tied to one end of the string, when its other end is rotated around a finger ?

(iv) What is the locus of the hands of a driver on the steering wheel ?

These are verbal illustrations based on observation. The teacher should devise many more to clarify the idea.

Some drawing exercises can also be set to give a clear idea of the locus. For example :

(i) If a point A moves to another point B having the same direction from A to B, it will trace a st. line AB. The st. line AB is the locus of the point.



(ii) If a point B moves at a distance of 1″ from another fixed point A, the path traced out by it will be the circumference of the circle with centre A and radius = 1″.

(iii) Take the case of a point P which moves at a distance of 1″ from a st. line AB. Because the point remains at the same distance from the straight line, the path traced out by it will evidently be parallel to the given straight line. So the locus of a point moving at a distance of 1″ from a straight line AB will be a pair of st. lines drawn ∥ to AB at a distance of 1″ from AB on either side of AB.

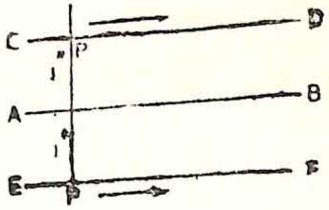


Fig. 74.

Let the students draw accurately a large number of points satisfying a certain condition, and by joining these points find the locus.

Thus they may draw exactly the locus of the mid points of a set of parallel chords in any closed curve.

This is a practical illustration of locus which should make the whole thing very clear to the learners. Every student can draw his own set of parallel chords of any closed curve of his choice.

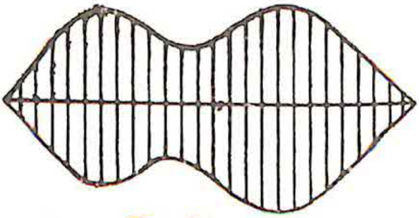


Fig. 75.

The second step is to show that the path of a moving point may be the place, location, of all points satisfying a given condition. Thus, if a point moves in a plane so as to be always at a given distance from a given point, the path is a circle. This circle is the place, path or location of all points in the plane which have the same distance from the fixed point.

The third step is to provide opportunities for determining loci from given conditions. These should be given by the use of simple constructions, general experiences of the child and the use of simple geometric models. The dynamic and static interpretations of locus should also be made meaningful to them.

DYNAMIC

It is an interpretation based on motion. For example "A point, moving in such a way that it is always equidistant from two given points, generates the perpendicular bisector of the line joining the two points."

AN EXAMPLE OF STATIC INTERPRETATION

The locus of points equidistant from two given points is the perpendicular bisector of the line joining the two points.

In proving any theorem on locus it is necessary and sufficient to prove the following things :—

(i) Any point on the supposed locus satisfies the given condition.

(ii) Any point not on the supposed locus does not satisfy the condition.

It may be explained like this : "If only the first thing were proved, there might be some other line in the locus; and if only the second thing were proved, the supposed locus might not be the accurate one."

HOW TO DETERMINE A LOCUS

Set the following exercises :—

(i) Locate several points which satisfy the given condition.

(ii) Draw a smooth line (straight or curved) or lines through these points.

(iii) Describe correctly the geometric figure representing the locus.

(iv) Prove that this is really the locus.

RIDERS

A rider to a theorem in geometry is the same as a problem on a formula in arithmetic. The exercises on a proposition enable the pupil to apply the proposition to different situations. The solving of riders provides real power in geometry. But learners mostly confine themselves to the learning of text-book propositions and omit exercises requiring thought. Some text-book writers also commit the blunder of giving printed instructions in the preface that exercises may be omitted on the first reading. Exercises are treated as something superfluous and beyond the comprehension of an average student. Even the teacher, in his anxiety to show a good pass percentage, guides his pupils to concentrate rather on propositions than on exercises. They think that propositions are more dependable. This practice seems to be based on the wrong notion that beginners cannot solve exercises, unless they have mastered a good number of propositions. So exercises are not regarded as important from the point of view of examination.

This view is erroneous and defeats the very purpose of teaching geometry. Mere mastery of readymade proofs of propositions cannot take the leaner very far. The development of the power of reasoning and systematic thinking will not be possible if exercises are not solved. The riders are the practice and application part of the propositions. The theorems will become clear only when they are used in solving riders. Knowledge is power only when it is applied.

Riders, therefore, have not to be ignored, nor should their solution be postponed to some later stage. The most opportune time for proving riders is immediately after a proposition has been proved. We should apply knowledge while it is still fresh, so that it is fixed in the mind simultaneously. The learning of theorems is rather passive and it is like watching some one else apply known facts, whereas rider work is active and the learner does the application himself. Only the solving of riders immediately after the proposition, gives the pupil a real sense of achievement and encourages him to do further work. The pupil's ability and progress in geometry should be judged by his ability to tackle exercises. Mere repetition of well-known facts does not make the pupil intelligent.

IMPORTANCE OF RIDERS

1. The rider work is very profitable for developing original thinking and power of application. Only original thinking represents true geometric work.
2. The subject can be introduced better by riders than by the study of readymade proofs of propositions. The solution of exercises can give the beginner a clear idea of the true nature of geometry and guards him against mechanical methods of study.
3. It is easier and interesting to attempt a simple rider than to understand a complicated proof of a proposition.
4. The riders have the quality of sustaining the interest of the pupil and preventing him from getting disgusted with the subject. The discovery of the proof of a rider is far more satisfying than memorising the set proof of a proposition.
5. The solution of a large number of riders after a proposition is a good means for recapitulation, understanding and clarity.
6. Riders can be much better graded than the text-book propositions. They are better representatives of the sequence and logic in order of difficulty. Their sequence can be made effective by the insertion of riders that lead from one theorem to another.
7. The capability, attainment and capacity of the pupils in geometry can be measured reliably by their ability to solve riders.

   In fact, the study of geometry should primarily be a course in the solution of originals.

HOW SHOULD THE RIDERS BE ATTEMPTED

GENERAL SUGGESTIONS :

1. First of all, the teacher has to acquire the ability to solve riders easily, accurately and promptly.
2. He should possess the knowledge of the various methods of attack.
3. He should always approach the matter heuristically. He should tell only when educing becomes difficult.

4. He should be well-versed in geometrical constructions also in order to meet the particular needs of the problem.
5. The statement of the rider should be carefully read to grasp the meaning of the words used. He must ascertain what is known and what is unknown. The 'if' and 'then' parts of the statement should be clearly analysed.
6. He should draw the most general figure possible, e.g., if it is about any triangle, draw the scalene triangle. An equilateral or right-angled triangle deceives the eye. Of course, accuracy is not to be neglected.
7. The pupil should be enabled to use 'Synthesis' and 'Analysis' and to combine the two. He may begin by assuming the truth of the theorem and see what follows from the assumption and then see if it can be proved true without the assumption, by reversing the process.
8. He may also begin by assuming the theorem to be false and then try to show the absurdity of the assumption.
9. In selecting a method of proof, the pupil should recall mentally the axioms, postulates, definitions, theorems and corollaries already proved to sort out the known methods which may in any way be helpful in solving the rider in hand.
10. While referring to some axiom, definition or theorem, a brief statement or enunciation must invariably be given.
11. The teacher should prepare, in advance, revealing questions on the riders in hand, and these questions should be used to provide clues to the students.
12. The teacher should inculcate in the pupil habit of asking himself :—"What do the data tell me ? Which theorem do I know which may enable me to proceed with the proof of the rider ? Is there any piece of data that I have failed to use ? Is there any necessity of completing the figure by some additional construction?

HELPFUL METHODS IN SOLVING EXERCISES—

METHOD I

The equality of lines and angles is usually proved by showing the congruency of the triangles containing those. This principle cannot be expected to apply in every case. The pupil should always resort to analysis by asking himself questions such as "What is the usual and possible method of demonstrating the equality of line and angles? He should always first try to find out the clue in this manner.

METHOD II

If the lines or angles where equality is to be shown are not contained in certain equal triangles, the student will have to complete the equal triangles by having additional constructions. The triangles have to be made, if not already provided. This is an important principle which finds an extensive use.

METHOD III

If it is not possible to prove the equality of the required pair of triangles, a via media has to be found by proving first the congruency of some other pair or pairs, whose equal parts will help to show consequently the congruency of the original triangles. The learner should not always think of jumping over to the final requirement; there may be certain intermediary requirements also.

METHOD IV

To prove that an angle is a right angle, it is usual to demonstrate that it is equal to its supplementary adjacent angle. The equality of adjacent angles may further depend on the congruency of two triangles containing them.

METHOD V

Parallelism of lines can be shown by means of equal alternate or corresponding angles.

METHOD VI

The equality of angles is also occasionally shown by the isosceles proposition.

METHOD VII

To prove that a st. line is equal to half of another line, either double the first or bisect the latter and prove them to be equal. The same relation between angles is proved in a similar way.

METHOD VIII

To prove that a st. line '*a*' equals the sum of two lines '*b*' and '*c*'. Construct the sum of *b* and *c* and prove it equal to '*a*' or construct the difference of '*a*' and '*c*' and prove it equal to '*b*'.

METHOD IX

Try one method. It that is not adequate, leave it and try another. Proceed in this manner.

A FEW LESSONS IN RIDERS

Ex. 1. In a $\triangle ABC$, $AB=AC=BC$, A is joined to D a point in BC, E is the mid pt. of AD. Prove that $EC > EA$.

To begin with, an equilateral triangle ABC, will be drawn, as obviously given AD is joined after taking any pt. D in BC AD is bisected at E, and EC is joined.

Given : What is the given part of the statement ?

$AB = AC = BC$ and $AE = ED$.

To prove :—What is to be proved ?

$EC > EA$.

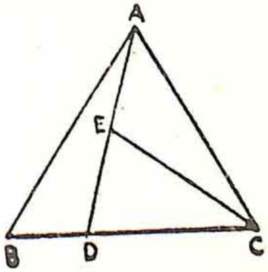


Fig. 76

Proof :—How can we prove a line greater than the other ? If they are the sides of the same triangle and the angle opposite the greater side is greater than the angle opposite the other.

We have to show $EC > EA$ or $EC > ED$
(Since E is the mid. point of AD)
Which is the triangle containing EC and ED ?
It is $\triangle$ EDC.
Which are the angles opposite to EC and ED ?
The angle opposite to EC is $\angle EDC$,
and the angle opposite to ED is $\angle ECD$.
What is to be proved now ? $\angle EDC > \angle EDC$.
What is $\angle EDC$ equal to ?
$EDC = \angle ABD + BAD$
(Since it is the exterior angle of another triangle ABD)
Then how to show $\angle EDC > \angle ECD$?
If $(\angle ABD + \angle BAD) > \angle ECD$

What is the part of 'given', we have not utilised so far ?

$AE = ED$ has been used, but the part $AB = AC = BC$ has not been used so far.

How can we use it ?

We should convert this property into the equality of the three angles, i.e. $\angle A = \angle B = \angle C$.

So try to fit it into the incomplete argument.

We can say, $\angle ABD = \angle ACD$.

$\therefore$ the problem is solved,
if $(\angle ACD + \angle BAD) > \angle ECD$, which is obviously true, because
$\angle ECD$ is a part of $\angle ACD$ and hence alone
$\angle ACD > ECD$.

The pupils can be guided to go back with the argument to say that $EC > ED$ Q.E.D

or $EC > AE$.

Ex. 2. If the diagonals of a ‖ gm. are equal, it is a rectangle. What is given in this case ?

Given : ABCD is a ‖ gm.

$AC = BD$.

But what are the qualities of this ‖ gm ?

$BC = AD$, $AB = CD$ and $\angle ABC + \angle DCB = 2$ right $\angle$s.

What has to be proved ?

To prove : ABCD is a rectangle.

What should a rectangle possess in addition to being a ‖ gm. ?

One of its angles should be 90^0

i.e. $\angle ABC = 90$ is to be proved.

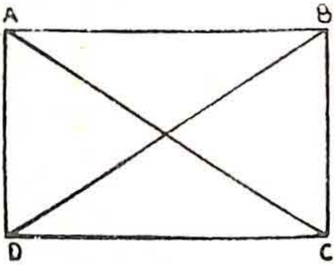


Fig. 77

Proof : How can we show that $\angle ABC = 90^0$

What part of the 'given' can be used for this ? We know that $\angle ABC + \angle DCB - 2$ rt. $\angle$s. if we can show these two $\angle$s to be equal, each of these will automatically be equal to 90^0. How to show the equality of these angles ?

By proving the congruency of two triangles containing them. Which are the triangles ? $\triangle$s ABC and DCE. What are the known equal elements of these triangles ?

$BC = BC$.

$AC = BD$. [The two diagonals are equal]

$AB = CD$. [$\because$ ABCD is a ‖ gm.]

$\therefore \triangle ABC = \triangle DCB$.

Our previons requirement is fulfilled i. e.,

$\angle ABC = \angle DCB$.

$\therefore \angle ABC = 90^0$

Consequently ABCD is a rectangle.

CONSTRUCTIONS

In geometry the words "to construct" mean to draw accurately. The only instruments permissible in these constructions are the straight edge for drawing straight lines and the compasses for drawing arcs and measuring lengths of lines. Efficiency in constructions requires skilful manipulation of these instruments. Actual doing is very necessary. A student may verbally know how to construct a certain geometrical figure, and yet may fail to draw it accurately, when actually required to do so. Lack of practical experience is generally responsible for this failure. Accuracy and neatness require good accurate geometrical instruments. Further constant practice is the only way to acquire self-confidence in construction work.

PARTS OF A CONSTRUCTION PROBLEM

(i) A statement of the problem.

(ii) A representation of the given parts.

(iii) A representation of the given parts in terms of the drawing.

(iv) A statement of what is to be constructed.

(v) The actual constructions, with a description of and relevant authority for each step.

(vi) A statement that the required construction has been made.

(vii) A proof of the construction.

DIRECTIONS FOR CONSTRUCTION WORK

1. The teacher should see that the pupils are properly equipped with an accurate and reliable set of geometrical instruments and fine, hard and sharpened pencils. Special drawing pencils should be used. Ordinary pencils do not give neat work.
2. All straight lines are to be drawn from left to right.
3. All lines should be of uniform thickness. The lines should be drawn as thin and sharp as possible.
4. Different steps of construction should be clearly shown.
5. The lines to be joined for proof should be dotted.
6. Rubbing is to be forbidden.
7. The exercise books should be placed on a hard flat surface.
8. The dot is unnecessary, when a point is fixed by the intersection of two lines.
9. The length of a straight line should not be measured directly from a ruler : an indefinite line should be drawn and from it the given length should be cut off with the help of compasses.
10. The pair of compasses should be held by the head, so that there are little chances of inaccuracy even by the slightest movement of its legs.
11. While drawing, the two legs of the compasses should be equal in length. Their joint should be screwed tight, if it is shaky.
12. It is often desirable to discuss the solution of a problem as

to the number of solutions, special cases, limitations and applications.

13. It may be often useful to draw the required figure by eye and then to study the properties of the figure in order to get ideas for proceeding with the construction.
14. Sketch (do not construct) the figure as it will appear when completed.
15. In the sketch, show the parts of the figure that are given.
16. Study the sketch to discover (a) which given part should be constructed first; (b) how the additional parts can be constructed with the help of known constructions and theorems.
17. After determining the method, construct the figure showing each step.
18. Every construction should be carefully checked by the pupil for its accuracy by measuring with ruler and protractor.
19. As soon as a new construction is taught, it should be applied to new situations.
20. When a known construction enters into a new construction it should always be made in full.

ANALYSIS IN CONSTRUCTION WORK

Like other aspects of geometry, construction work should also be attempted analytically as far as possible. Here the teacher starts from the result and works backwards. By this procedure he reaches a simpler theorem already known to be true or arrives at some result which gives a clue to the necessary construction. The reason behind every step of construction must be adequately grasped by the pupil.

The following example will illustrate the use of 'analysis in construction problems :—

PROBLEM

Construct a ∥ gm equal in area to a given triangle having one equal to a given angle.

Analysis :—A ∥ gm. is to be constructed.
What do we need to construct a ∥ gm. ?
(Two sides and an angle or one side, height and an angle etc.)
What is available in the enunciation ?
(Only an angle is available).
What is the area of a triangle equal to ?
($\frac{1}{2}$ × base × height).

What is the area of a ∥ gm. equal to ?
(base × height).

What should be the relation of the base and height of the required ∥ gm., with the base and height of the given triangle?[Two possibilities :—(1) If base of ∥ gm. = the base of the △, then height of the ∥ gm, should be = $\frac{1}{2}$ of the height of the △. (2) If base of ∥ gm = $\frac{1}{2}$ of the base of the △, then height of the ∥ gm. should be = the height of the △.]

So the base and the height of the ∥ gm are respectively related to the base and the height of the △ in this manner.

Let the student then take the base of the ∥ gm. by bisecting the base of the triangle and then take its height equal to the height of the triangle by drawing a line ∥ to the base line through the vertex of the triangle. The next step will be to draw the second line of the ∥ gm by measuring the angle equal to the given angle. The ∥ gm. can be completed in this manner.

EXERCISES

1. What are the aims of teaching geometry in schools ? How far these aims are being achieved and what are the difficulties in the way ?

2. What are the different stages of teaching geometry ? Illustrate them by taking examples from the subject-matter ?

3. How will you improve the teaching of geometry in schools ?

4. How to teach the following in geometry ?

   (i) Axioms and postulates.
   (ii) Definitions.
   (iii) Propositions.

5 What are the different kinds of proofs in demonstrative geometry ? Illustrate them with actual examples ?

6. How should the theorems and riders be learnt by a student of geometry ?

7. How will you teach the following topics ?

   (i) Angle-sum of a triangle.
   (ii) Congruency of triangles.
   (iii) Similarity.
   (iv) Locus.

8. What would be your distinctive approach in the teaching of constructions in geometry ?

CHAPTER TWENTY

THE TEACHING OF TRIGONOMETRY

WHY TO TEACH IT

TO MANY PEOPLE, trigonometry is a college subject. This is quite true, because till recently it was never offered in the high schools to the extent that algebra and geometry were; whereas it is universally offered and often required in college because it is indispensable to the study of analytic geometry, calculus and other advanced courses in mathematics, engineering and the physical sciences. Perhaps the biggest change in school mathematics is due to dethronement of Euclid which took place about 1903. His 150 propositions have been replaced by about 60 formal theorems. To utilise the time thus saved, introduction of new ideas like trigonometry became natural.

The advent of the Higher Secondary system has gone a long way in dispelling the restricted view of the subject. There has come about the realisation that, while certain parts of trigonometry are indeed difficult and suitable only for relatively mature students, there is much, on the other hand, which is quite simple for and can be easily understood by students of higher secondary stage. It has been observed that some of the elementary concepts of trigonometry are more easily comprehended than much of the usual work in algebra. Also the application of these principles is quite interesting to the pupils in the final years of the secondary school. Such application offers an excellent means of correlating arithmetic with certain parts of informal geometry and with the solution of simple linear equations. The early introduction and simple treatment of these aspects of trigonometry may be expected to have considerable iufluence in removing the reluctance of student to undertake later the systematic study of the subject. Therefore, because of its general educational value and its initiating force, some preliminary work in trigonometry is almost necessary in a higher secondary school. It has to be admitted that the general maturity of the pupils at this stage is, inadequate for a systematic study of the subject. Consequently, its systematic study has to be deferred to the first year of college work.

WHAT PARTS OF TRIGONOMETRY SHOULD BE TAUGHT

The parts to be included must find their justification in the direct contribution which they can make to the goals of general education at this level. These may be thought of as potential practical values and as general educational values. The subject-matter should include : extension and clarification of the concept of ratio ! the use of symbolism; appreciation of the power and application of indirect measurement; understanding of the methods of accomplishing such measurement through the use of the right triangle and the tangent, sine. and cosine ratios; co-relation of ideas and procedures drawn from arithmetic, algebra and geometry and stimulation of interest in mathematics as a whole. Broadly speaking, the teaching of trigonometry should begin long before the formal introduction of this branch. The students should be acquainted with instances dealing with indirect measurement of distances. They should be allowed to undertake actual field projects. It is necessary that they have clear concepts of the meaning of similar figures (especially similar triangles); of a ratio as a comparison of two quantities in the sense that one is a certain fraction of the other; and also of a ratio as a single number. They also must understand the particular meanings of the tangent ratio, the sine ratio, and also know how to use the table of natural functions. The meaning of such terms as "angle of elevation" and 'angle of depression" should also be clear to them. They have to be given practice in analysing problems, in making working drawings, in selecting the appropriate functions to use, in setting up and solving equations involving these functions, and in substitution and evaluation.

DEVELOPING THE MEANING OF A TRIGONOMETRIC FUNCTION

The first thing to be done is to make clear the meaning of the trigonometric functions as ratios and as numbers. Probably the best way to accomplish this is to have the students actually make careful measurements of the angles and sides of right triangles and compute the numerical values of the ratios representing different functions.

It is a mistake to start by giving the pupils the six ratios at once. It would be better to do the cosine first, then the sine, and then not to introduce any other ratio for some days. It is easy to show that the cosine and sine of any acute angle are each less than 1; so they will easily remember cosine $=\dfrac{\qquad\qquad}{\text{hypotenuse}}$, sine $=\dfrac{\qquad\qquad}{\text{hypotenuse}}$. If they are told that the two sides for the cosine contain the angle, they can easily write down sines and cosines from right-angled triangles in various positions. For obvious reasons, this work is limited to situations that involve functions of acute angles. As soon as they are thoroughly familiar with the sine and cosine, they may be taught that $\text{tangent} = \dfrac{\text{sine line}}{\text{cosine line}}$, $\text{cotangent} = \dfrac{\text{cosine line}}{\text{sine line}}$. They may be more specifically defined in terms of the sides of a right triangle. Let angle BAC be the given angle.

$$\text{Cosine } \angle BAC = \frac{\text{Adjacent side}}{\text{Hypotenuse}} \text{ or } \frac{AB}{AC}.$$

$$\text{Sine } \angle BAC = \frac{\text{Opposite side}}{\text{Hypotenuse}} \text{ or } \frac{BC}{AC}.$$

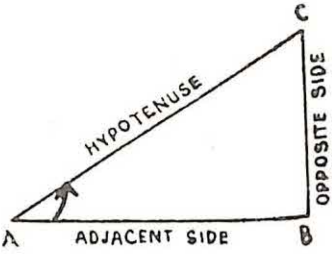


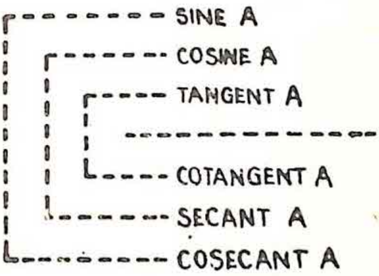
Fig. 78

$$\text{Tangent } \angle BAC = \frac{\text{Opposite side}}{\text{Adjacent side}} \text{ or } \frac{BC}{AB}.$$

An interesting and valuable exercise can now be introduced by having the pupils make measurements and computations necessary for the compilation of a table of sines, cosines and tangents for the acute angles, say, which are integral multiples of 5 degrees. For each of the angles considered, at least two or three students should make determinations of the functions for purposes of comparison, checking, and averaging of the results.

It may be made clear, that the functions are, after all, merely definitions of certain ratios. After having learned the meanings of the sine, cosine and tangent, the students should have no difficulty in understanding the meanings of cotangent, secant and cosecant.

The student will perhaps be curious to know why the names of the other three except for the prefix "Co". It should be explained that the cosine of an angle is the sine of the complementary angle which is the other acute angle in the right triangle, and that in the word 'cosine' co-stands for "complement." Similar explanations can be given about the corresponding relation of the cotangent to the tangent of an angle, and of the co-secant to the secant. Let the students learn to recite names of the six functions in the usual order, i.e., sine, cosine, tangent, cotangent secant and cosecant. If these names are then written in a vertical column as in the diagram it is apparent that the two functions joined by any one of the three brackets are reciprocals of each other. The device specifies and emphasises the reciprocal relationships.



Inverse Functions : They appear to be difficult to the students due to the fact that their meanings have not been adequately explained. Also the notations $\sin^{-1} \frac{1}{2}$, $\cos^{-1} 1$, etc., are sometimes confusing in the beginning because of the students' tendency to associate this notation with the idea of an exponent. Great care should be taken to ensure that the students will realise that the symbols $\sin^{-1} \frac{1}{2}$, $\cos^{-1} 1$, etc., represent angles rather than functions of angles. It should be emphasised that the symbols for the inverse functions are to be read "the angle whose sin is $\frac{1}{2}$" and "the angle whose cosine is 1", respectively. A systematic drill should be given in writing down given functions and the corresponding inverse functions in parallel columns, thus :

$$\sin 30^\circ = \tfrac{1}{2} \qquad\qquad \sin^{-1}\tfrac{1}{2} = 30^\circ$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} \qquad\qquad \cos^{-1} \frac{1}{\sqrt{2}} = 45^\circ$$

$$\text{Secant } 0^\circ = 1 \qquad\qquad \text{Secant}^{-} 1 = 0^\circ \text{ etc. etc.}$$

This drill may be very short but should be given at regular intervals throughout the course.

TEACHING THE FUNCTIONS OF SPECIAL ANGLES

The special angle O, 3, 30, 45, 60 and 90 degrees are of great importance and occur so frequently that the students should make them a part of their ready knowledge. Memorising them by heart should not be a burden. But they can also be immediately and easily derived from three elementary geometric considerations.

Consider the right triangle ABC with an acute angle A of 45° and a right angle at B. From the illustration it can be seen at once that such a triangle is formed when a diagonal of a square is drawn. If the length of one side of the squre is taken as one unit, each leg of the triangle will be 1 unit in length and the diagonal of the square, which is the hypotenuse of the triangle, will be found by the pythagorean theorem to have a length of $\sqrt{2}$ units. With these numbers known, all the functions of 45 degrees can be written down immediately. The only quantities needed are the lengths 1, 1, $\sqrt{2}$. Also there is no need to memorise them as they are quite apparent. The functions are as follows :

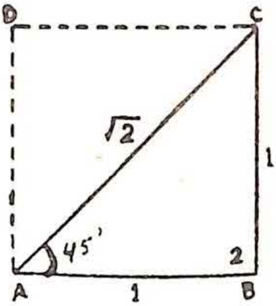


Fig. 79

$$\sin 45^\circ = \cos\ 45^\circ = \frac{1}{\sqrt{2}}\ ;\ \tan 45^\circ = \cot 45^\circ = 1$$

$$\text{cosec}\ 45^\circ = \sec\ 45^\circ = \sqrt{2}$$

Consider now an equilateral triangle ∠ADC with A bisected as shown. Thus there is a 30—60 degree right triangle ABC, whose hypotenuse AC is twice as long as the leg BC. If the length BC is treated as one unit, then AC has a length of 2 units, and by pythagorean theorem, $AB = \sqrt{3}$ units. So, from the figure.

$$\sin 30^\circ = \tfrac{1}{2} = \cos 60^\circ$$

$$\cos\ 30^\circ = \frac{\sqrt{3}}{2} = \sin 60^\circ.$$

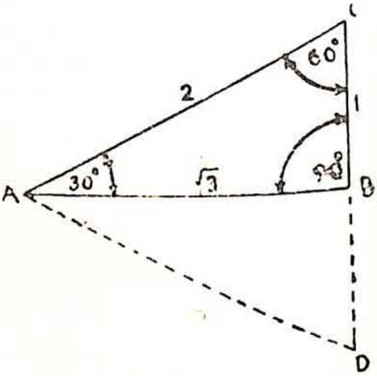


Fig. 80

$$\tan\ 30^\circ = \frac{1}{\sqrt{3}} = \cot 60^\circ$$

$$\cot 30^\circ = \sqrt{3} = \tan 60^\circ.$$

$$\sec\ 30^\circ = \frac{2}{\sqrt{3}} = \text{cosec}\ 60^\circ$$

$$\text{cosec}\ 30^\circ = 2 = \sec 60^\circ$$

A most effective device for studying the functions of 0 and 90 degrees is the unit circle.

The functions of all the special angles can be consolidated in the form of a table on a chart as :

S	0°	30°	45°	60°	90°
SINE	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
COSINE	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
TANGENT	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
COTANGENT	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
SECANT	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
COSECANT	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Fig. 81

These values, except infinity, can be readily translated into decimal equivalents if desired. The student who learns to derive them with the help of special triangles and unit circle in this way need never be afraid of forgetting them, because he can reconstruct them at will. When they are associated with the triangles, they will carry for him a real functional meaning and will not be mere arbitrary and unrelated qualities.

FUNCTIONS OF THE GENERAL ANGLES

The subsequent transition to the function of the general angle involves a generalisation of the definition of the functions. But if it is explained carefully, there should be no difficulty.

The students have learnt to define the functions of an acute angle in terms of the sides of a right triangle, as—

$$\sin A = \frac{\text{Opposite side}}{\text{hypotenuse}}$$

$$\cos A = \frac{\text{Adjacent side}}{\text{hypotenuse}}$$

$$\tan A = \frac{\text{Opposite side}}{\text{adjacent side}}$$

Now if a set of perpendicular coordinate axes is drawn with origin at O and the positive X-axis passing through B, the function of the acute angle A can be generalised in terms of the abscissa, ordinate and distance of C, or any other point on the terminal side of the angle. That is, the "opposite side" in the triangle is now y, the ordinate of C; the "adjacent side" is x, the abscissa of C; and the "hypotenuse" is r, the distance of C.

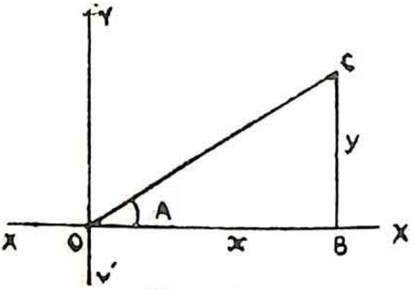


Fig. 82

Consequently the functions can be redefined as :

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{\text{ordinate}}{\text{distance}} = \frac{y}{r}.$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{\text{abscissa}}{\text{distance}} = \frac{x}{r}.$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x} \cdot \text{ and so on.}$$

One thing is especially noticeable. The teacher has so far redefined the functions only for acute angles. What shall he do in the case of angles greater than 90 degrees or in the case of negative angles ?

He can disclose to the students that these new definitions provide them a concept, a nomenclature, a symbolism that will apply as readily to angles of any size, whether positive or negative, as it does to acute angles. These definitions will be acceptable for the purpose of defining the functions of any angle, and will ensure uniformity and convenience. The student will only have to recall the "standard position" of the angle with reference to the coordinate axes; i.e., its vertex will always be considered to be at the origin, its initial side the positive X axis, and the position of its terminal side determined by the description of the angle or the conditions of the problem.

To explain the transformation of the functions of special angles into the functions of general angle, let A be the given angle in standard position as shown in the figures.

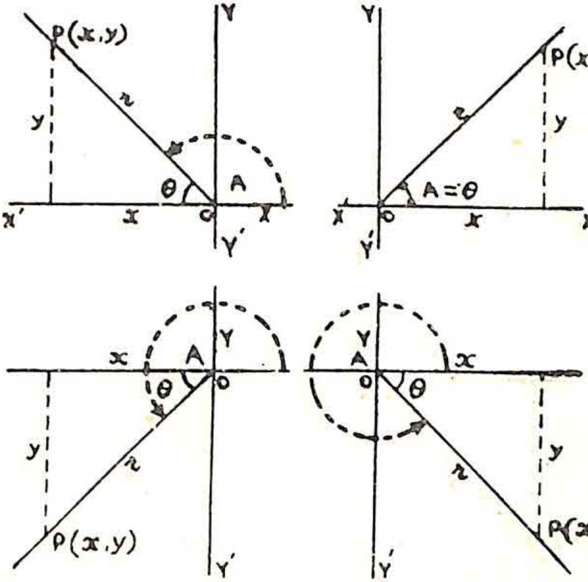


Fig. 83

If any point P (x, y) is taken on the terminal side of angle A, and a perpendicular is dropped on the X-axis, there will be formed a right triangle. That will be called the "triangle of reference". The values of the functions of A are determined by the lengths of the sides of this triangle which are respectively x, y and r.

Now, if A is known, the student can also find at once the angle θ, without regard to sign, the smallest angle which OP makes with the X-axis. The value of this angle will never exceed 90°. The values of functions of A will be the same as the values of the functions of θ except for sign. The functions of θ can be found from the tables, and the signs of the corresponding functions A can be determined by noting the signs of x and y (r is always positive) in the triangle of reference.

It is good for the students to work a number of examples on the basis of the above illustration.

TEACHING THE USE OF TRIGONOMETRIC FUNCTIONS

As soon as the students have comprehended the meaning of the tangent ratio, the teacher should show how it is used in finding distances or angles without direct measurement. Thus such problems as the following may introduce the pupils aptly to the application of the functions.

(i) From a point 50 feet from the base of a tree, the angle or elevation (A) of the top of the tree is found to be 29 degrees. Find the height of the tree ?

(ii) A ladder 25 feet long is placed against a vertical wall so that it makes an angle of 57° with the ground. At what height above the ground does the ladder touch the wall ?

In this subject there are many applications which should be made known to the pupils to intensify their interest. A large share of problems in trigonometry textbooks are problems which involve the determination of distances, angles, or area in relation to land measurements; in other words, problems typical of those which the surveyor encounters in his work.

Though less known, there are many interesting applications of trigonometry in engineering, astronomy, geology, navigation and aviation.

The following table, extracted from the report of an analyst shows the frequency of occurrence, the percentage of occurrence, and the number of trigonometric formulas found in the textbooks in various fields :

Table of occurrence of Trigonometric Formulae in Twenty Textbooks.

	Occurrence	Per cent	Number of formulae
Five higher mathematics texts	314	36.7	41
Five texts in mechanism	299	35.0	23
Five texts in surveying	125	14.6	19
Five texts in electricity	118	13.7	13
Total	856	100.0	

Instances can be multiplied. Here are a few :

For a coil rotating uniformly in a magnetic field, the instantaneous induced voltage (E_i) is given by the formula $E_i = E_m \sin \theta$, where E_m is the maximum voltage and θ is the phase angle.

The current I (amperes) flowing through a tangent galvanometer having n turns, of radius r, in the earth's field H, produces an angular deflection θ. The formula for the determination of the amount of current is

$$i=\frac{Hr}{2\pi n}\tan\theta$$

The index of refraction of light, n, is given by the formula $n=\frac{\sin i}{\sin r}$, where i is the angle of incidence, and r is the angle of refraction.

The motion of projectiles is another general problem involving trigonometric formulas. For projectiles having an initial velocity v at an angle θ with the horizontal, the following formulas hold.

Maximum height $h=\frac{v^2\sin^2\theta}{2g}.$

Time to maximum height $t=\frac{v\sin\theta}{g}.$

Total time of flight $T=\frac{2v\sin\theta}{g}.$

Horizontal range $R=\frac{v^2\sin 2\theta}{g}.$

General equation of trajectory $y=x\tan\theta-\frac{gx^2}{2v^2\cos^2\theta}.$

One of the basic problems of automatic engineering is the determination of the force P needed to balance a weight W on a plane inclined to the horizontal at a given angle α. This relationship is expressed by the formula $P=W\sin\alpha$.

Two important problems of aviation are the determination of the wind-correction angle and the determination of ground speed. Trigonometric formulae are applicable in them. If, for example, the wind correction angle is designated by x and the angle between the wind direction and the true course by y, then

$$\frac{\text{Wind velocity}}{\text{Air speed}}=\frac{\sin x}{\sin y}.$$

If the same notation is used, the formula for determining ground speed is as follows :—

$$\frac{\text{Ground speed}}{\text{Air speed}}=\frac{\sin(y-x)}{\sin y.}$$

These miscellaneous illustrations of the use of trigonometry may serve to suggest others. Such applications rarely fail to increase interest of the subject. These will not only enhance the interest of the students, but broaden the teacher's horizon and add zest to his teaching also.

TEACHING THE FUNCTIONS OF TWO ANGLES

The students will need to use the functions of the sum or difference of two angles, of a double angle, or of a half angle. He may also have to use certain other formulae involving two angles, so it is necessary that these formulae be developed in terms of the original angles and listed for reference. It is not necessary to memorise all of these formulae. It is desirable, however, that the pupils see and understand the ways in which the formulae are developed and also memorise a few formulae which are fundamental and of frequent use. The following may be listed as the fundamental formulae of this kind :

$$\sin\ (A\pm B) \qquad \cos\ (A\pm B \qquad \tan\ (A\pm B)$$
$$\sin 2A \qquad \cos 2A \qquad \tan 2A.$$

The development of a large number of the formulae of this group is an excellent exercise in the application of algebraic processes to trigonometric functions. After the formulae $\sin\ (A+B)=\sin A \cos B+\cos A \sin B$ has been proved, all the rest of these formulae may be derived from this one through the application of simple algebraic and trigonometric techniques. For example:

1. $\sin (A-B)=\sin [A+(-B)]$
$$=\sin A \cos (-B)+\cos A \sin (-B)$$
$$=\sin A \cos B-\cos A \sin B.$$
2. $\cos (A+B)=\sin\ [(90°-(A+B)]$
$$=\sin [90°-A)-B]$$
$$=\sin (90°-A) \cos B-\cos (90°-A) \sin B.$$
$$=\cos A \cos B-\sin A \sin B.$$

It is strange that in the teaching of these formulae this fact is almost never emphasised. The student will be fascinated on seeing that all these stem from a single basic formula.

This basic formula should be established geometrically and then verified by different concrete examples such as—

$$\sin 90°=\sin (60+30°)=\sin 60° \cos 30°+\cos 60° \sin 30°.$$

VARIATION OF THE FUNCTIONS

It is very interesting for the pupils to note that any change in the size of angle is accompanied by a corresponding characteristic change in each of the functions of the angle. This fact should be repeatedly emphasised by the teacher and should be illustrated by all available means and devices. One such means is a thoughtful examination of the tables of functions. It should be specifically pointed out that the values, written in the tables, merely represent particular stages in a continuous variation and that, between any two successive values given, there exist innumerable intermediary values.

Just as the concept of variation enriches the study of other branches of mathematics, so it does the study of trigonometry. The concept can be strengthened by certain mechanical devices. Perhaps the most commonly used and most effective device is the construction and study of graphs of these functions.

A real understanding of these graphs elucidates several important things about this variation. In particular, the following considerations are worthy of note.

1. There is a general similarity in the shape of the graph of any function and its corresponding "cofunction".
2. Sin θ and cos θ are always finite and continuous over the entire range of values of θ, while for certain values of θ the other functions become infinite and have points of discontinuity.
3. The sine or cosine of an angle can never be greater than $+1$ or less than -1, while the secant and cosecant can never have values between these bounds. Numerical illustrations of these facts can be given.
4. From the graphs it will be found that, when the angle is given by $\left(k\pi+\frac{\pi}{4}\right)$ for all integral values of k, any function of the angle is equal to its corresponding cofunction. The students may be asked to find out the explanations.
5. The graph of any function, say sin θ, is also the graph of the inverse function. The distinction lies merely in the choice of the independent variable.

Almost all textbooks contain graphs of the functions for reference; in many cases the students are required to construct their own graphs. Sufficient time must be devoted to the interpretation of graphs and the discussion of their implications.

The Solution of Triangles

The trigonometric solution of the right triangle, already discussed in detail, is quite a simple affair. The solution of the oblique or general triangle, however, is not so simple. It requires the development of numerous special formulae.

There are "the four cases" of the general triangle :

Case 1. Given two angles and any side.
Case 2. Given two sides and an angle opposite one of them.
Case 3. Given two sides and the included angle.
Case 4. Given the three sides.

The most fundamental of the various special formulae are the law of sines, the law of cosines and the law of tangents. The mastery of these and many similar formulae and their use cause a good deal of difficulty. The commonly experienced difficulties are :—

1. Difficulty in following and understanding the derivation of the formulae.
2. Difficulty in learning by note the various formulae and then keeping them in mind without confusing them.
3. Difficulty in knowing what formulae to use in a particular case.
4. Difficulty due to lack of systematic pre-planning and layout of written work.
5. Difficulty in checking the solution.

The development of the fundamental trigonometric laws should be done by pointing out the way, by laying out the main steps in the development in quick bold outline just as one lays out a plan of proof for a proposition in geometry, before proceeding to write down the complete synthetic proof in finished form. The emphasis should be on the understanding of the main plan whereby the proof can be reconstructed without being crammed. In some cases it is necessary to draw upon previously developed formulae.

The teacher should give the students criteria for determining which or when formula is to be used. One criterion is as follows : The formula must be one which expresses an unknown part of the triangle completely in terms of the given parts. If he is required to do this a number of times, he will soon learn that the law of sines is the appropriate formula to use, given (1) two angles and any side or (2) two sides and an angle opposite one of them: and that the law of cosines will enable him to solve triangles in which the given parts are (1) two sides and the included angle, or (2) the three sides. The association of the appropriate formulae with the particular cases would be strengthened if each formula were developed when the case requiring it were taken up and the formula then immediately applied.

The following example is illustrative of the orderly procedure in solving such cases. Though layouts will naturally vary in detail according to the conditions and the requirements of the problem, yet the procedure given below is perfectly general and can be adapted as needed.

Given $a=81.6$
$b=121.5$
$C=32°18''$
Find A,B and C.

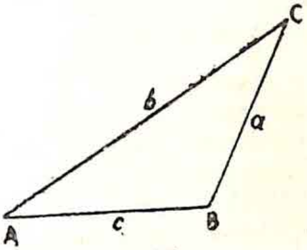


Fig. 84

SOLUTION

Formulae :

$$B+A=180°-C$$

$$\left.\frac{1}{2}(B-A)=\frac{b-a}{b+a}\tan\frac{1}{2}(B+A)\right)$$ to find A and B.

$$c=\frac{a\,\text{Sin}\,C}{\text{Sin}\,A}$$ ...... to find c.

$$c=\frac{b\,\text{Sin}\,C}{\text{Sin}\,B}$$...... to check the work.

TEACHING RADIAN MEASURE

It requires a good understanding of the functions of the general angle. To enable the pupils to acquire a thorough understanding of the definition and meaning of a radian, the geometrical explanation and illustration should be given repeatedly. They should be led to form a clear mental picture as shown in the figure. This will give a concrete basis for thinking in terms of radians. It will provide a means for recalling instantly why the angle π is the same as 180 degrees: why $\frac{\pi}{2}$ is the same as 90 degrees; why $\frac{\pi}{3}$ is the same as 60 degrees; why 2π is the same as 360 degrees; etc. With this

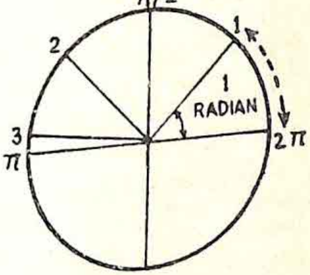


Fig. 85

sort of picture in mind, the student should have no need to memorise the relations between degrees and radian measures.

After they have acquired an understanding of the meaning of radian measure, it is useful for them to have some special practice in determining the degree equivalents of angles expressed in radians, and the radian equivalents of angles expressed in degrees. In this connection it may be desirable to have the degree value of one radian memorised. For the special angles O, $\frac{\pi}{2}, \frac{\pi}{3}$, π, 2π, 45 degrees, 120 degrees etc., the student should soon become able to give the corresponding equivalents at sight. Similar practice in inverse functions should be given, the principal values of the angles being given both in radians and degrees. The students should be made to understand, that, in expressing angles in terms of radian measure, it is customary to omit the word "radians". They should be able to realise while reading such expression as sin $\frac{\pi}{3}$, tan 2π cos $\frac{2\pi}{3}$, etc., that the angles are given in radians rather than in degrees. It may be pointed out that in higher mathematics radian measure is used almost exclusively.

TEACHING TRIGONOMETRIC EQUATIONS AND IDENTITIES

Confusion often exists in the pupil's mind as to the distinction between identities and equations of condition. This confusion should be cleared up in the very beginning.

A trigonometric equation is an equation involving one or more trigonometric functions of the unknown angle and is true only for particular values of this unknown. In contrast to such conditional equations there is another class of trigonometric equalities which are true for all values of the unknown. These are known as trigonometric identities. From this, it is clear that in dealing with trigonometric equations, the job is to solve the equation; i.e., to find the value, or values, of the unknown which will satisfy the equation. On the other hand, the problem in dealing with identities is to prove that they actually are identities; to show that the two members can be reduced to the same expression or expressions which are recognised as being identical.

Trigonometric equations really are equations, and therefore are subject to treatment under the algebraic laws of the equation besides being subject to the algebraic transformation of either member separately or to the trigonometric transformation of the trigonometric elements involved. For example,

$\sin 2x = 2 \sin x \cos x$

$\tan x = \dfrac{\sin x}{\cos x}$.

$\therefore\ 2 \sin x \cos x = \dfrac{\sin x}{\cos x}$

$2 \sin x \cos^2 \quad = \sin x$

$2 \sin x \cos^2 x - \sin x = 0$

$\sin x\,(2\cos^2 x - 1) \quad = 0$ (left member factorised)

This can be true only (1) if $\sin x = 0$ or (2) if $2 \cos^2 x - 1 = 0$

$\therefore$ the equation is true for the values

(1) $x = 0°$ or $180°$

(2) $x = 45°, 135°, 225°$ or $315°$

and for no other values of x wnere $x \leq 360°$.

Note that the algebraic transformations which have been employed are all entirely legitimate since we started with an admitted equation and sought merely the solution of this equation.

In practically all textbooks the trigonometric identities proposed for proof are stated in the form of equations. This creates a wrong tendency in the students to treat these identities under the algebraic laws of the equation. But this is not legitimate. Consider, as a case, the following illustration:

$$\text{Prove} \quad \frac{2-\cos x}{\cos x} = 2 \sec x - 1 \qquad \text{(A)}$$

Suppose that this was regarded as an equation and treated as such in the following manner : multiplying on both sides by $\cos x$, we get

$$2 - \cos x = 2 \sec x \cos x - \cos x \qquad \text{(B)}$$

or $2 - \cos x = 2(1) - \cos x$.

This evidently gives an identity; but it is not a proof that $\dfrac{2-\cos x}{\cos x}$ and $2 \sec x - 1$ are identical. Rather, it is a proof that, if equation (A) is an identity, then equation (B) is an identity.

The fallacy involved in this method of reasoning may be shown even more clearly by taking a case which is evidently absurd on the face of it but which illustrates precisely the point in question. Let it be required to find whether or not $\sin x = -\sin x$ is an identity. By treating this is an equation and squaring both sides, we do get an identity, which is $\sin^2 x = \sin^2 x$. The reasoning would lead us to conclude that, since this is true, therefore the original pair of expression ($\sin x$ and $-\sin x$) must also have been identical. Obviously, however, we should not be justified in such a conclusion because the equation $\sin x = \sin x$ is not true for all values of x. The above technique establishes a valid proof of an identity, only when it is reversible.

In proving two trigonometric expressions identical, we may legitimately work with only one of them at a time. Either may be changed through legitimate operations of trigonometric transformation, factorisation, multiplication of factors, collection of terms, or substitution, into the form of the other, or both may be changed separately into some third form.

The general difficulties experienced by the students in working with trigonometric equations and identities are: (1) lack of systematic approach, and (2) lack of familiarity with the fundamental identities. Here are some suggestions to plan and carry out the work.

1. If an identity involves functions of two angles, transform them so that they are expressed in terms of one angle. For example, in proving that $\text{Sin } 2\theta = 2 \text{ Sin } \theta / \text{ Sec } \theta$ one should start by expressing $\text{Sin } 2\theta$ as $2 \text{ Sin } \theta \text{ Cos } \theta$.

2. All the functions involved should be expressed in terms of a single function by means of the fundamental identities. For example, in solving the equation $\text{Cos } x + \text{Sec } x = \frac{5}{2}$, the first step would be to express $\text{Sec } x$ as $\frac{1}{\text{Cos } x}$

3. Having reduced all functions to terms of a single function, all the terms involving one function should be transposed to one side of the equation.

4. Having solved an equation for the value of some one function of the unknown angle, it may be possible to determine the angle by inspection and consideration of the values of the functions of special angles.

5. In proving an identity, try to convert one of the terms into an expression identical with the other.

6. If it appears that the fifth case is not possible, try to transform both members separately into some third identical expression.

7. If no previous methods is applicable, try to obtain an identity by transforming all functions involved into terms of the sine and cosine.

Success in solving trigonometric equations or in proving identities depends on close familiarity with the relation among the functions. The eight fundamental identites are:—

(1) $\mathrm{Sin}^2x+\mathrm{Cos}^2 = 1$ (5) $\mathrm{Cot}\ x= \dfrac{\mathrm{Cos}\ x}{\mathrm{Sin}\ x}$

(2) $\mathrm{Sec}^2x-\tan^2x = 1$ (6) $\mathrm{Cot}\ x= \dfrac{1}{\mathrm{Tan}\ x}$

(3) $\mathrm{Cosec}^2x-\mathrm{Cot}^2x = 1$ (7) $\mathrm{Sec}\ x= \dfrac{1}{\mathrm{Cos}\ x}$

(4) $\mathrm{Tan}\ x= \dfrac{\mathrm{Sin}\ x}{\mathrm{Cos}\ x}$ (8) $\mathrm{Cosec}\ x= \dfrac{1}{\mathrm{Sin}\ x}$

These have to be thoroughly memorised. The pupil should be already familiar with the formulae (6), (7) and (8) since they are merely the reciprocal relations pointed out in connection with the definition of the functions, and his attention should be called to the fact that (2) and (3) follow immediately from (1). In addition to these, the student should also be familiar with the functions of the double angles.

"The function hexagon' provides a very interesting aid to memorise all the above fundamental identities (except the first three of the list) and also the functions of double and half angles. These are the following interesting and useful conclusions about this hexagon.

1. The two functions at the ends of any diagonal are reciprocals of each other.

2. Any function is the product of the two functions between which it lies. Thus Cot x=Cos x × Cosec x; Cos x=Sin x. Cot x; etc.

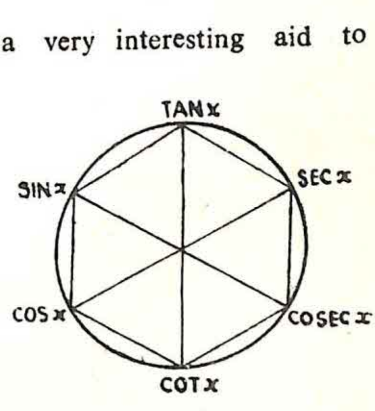


Fig. 86
The function hexagon.

3. Any function can be expressed as a fraction in which the numerator is either of the adjacent functions and the denominator is the one beyond that. Thus.

$$\mathrm{Sin}\ x= \frac{\tan x}{\mathrm{Sec}\ x};\ \mathrm{Cot}\ x= \frac{\mathrm{Cosec}\ x}{\mathrm{Sec}\ x}\quad \text{c.}$$

4. The product of any three alternate functions is 1.

Thus Cos x tan x Cosec $x=1$ and

Sec x Cot x Sin $x=1$.

This mechanical device gives 26 separate identities for reference. Further, it is so simple, economical and powerful that students will find it extremely helpful in their work with trigonometric identities and equations.

EXERCISES

1. Why should trigonometry be taught as a part of mathematics in schools and which of its topics can be usefully taught ?
2. How will you develop the meanings of trigonometric functions in case of beginners ?
3. How will you teach the following to the beginners ?
   - (*i*) Use of trigonometric functions.
   - (*ii*) Variations of the functions.
   - (*iii*) Radian Measure.
4. How will you introduce the teaching of trigonometric equations and identities ?

CHAPTER TWENTY-ONE

THE TEACHING OF SOLID GEOMETRY

WHAT IS SOLID GEOMETRY

SOLID GEOMETRY TREATS of figures and bodies in space. It is a study in three dimensions: (i) Length (ii) Breadth; and (iii) Thickness.

A point has no dimension, i.e., it has no length, no breadth and no thickness. A line has one dimension, i.e., it has only length, no breadth and no thickness. A surface has two dimensions, i.e., it has length and breadth, and no thickness. A solid has three dimensions, namely length, breadth and thickness.

ITS INTRODUCTION

The introductory ideas and concepts of solid geometry can easily be explained through concrete material. A plane can easily be presented with the help of a plane surface of a book, a plane table top and a perfectly even floor. It is perfectly flat and even like the surface of still water, or like a perfectly levelled playground. The formal definition of 'a plane' should come after a number of visible and obvious experiences. This should also be derived by comparnig a plane surface with a curved one. If the student takes two points in a curved surface, the st. line joining them will not lie wholly in that surface, whereas the st. line joining two such points in a plane surface will lie wholly in that surface. The idea of the figure of a plane should be developed from the student's knowledge of drawing. It is on the principles of drawing that a plane is represented by a parallelogram. ABCD is a plane. Two pts. L and M are taken in it and joined. The straight line

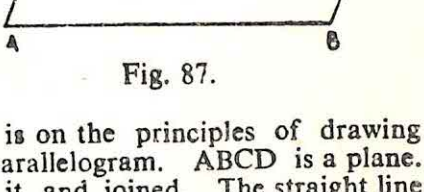


Fig. 87.

LM (produced both ways) will lie wholly in the surface of the plane.

Similarly the idea of parallel planes can be given with the help of the opposite walls of the class-room. Planes are parallel, if they do not meet when produced both ways to any length. Two adjacent walls of the class-room are intersecting planes. Two planes intersect when they have a point in common. There must be at least two points common to both of them. Similarly, lines drawn on the same plane are coplanar lines. The lines on a page of exercise book are an illustration. These lines are also parallel lines. They are coplanar; they do not meet if produced both ways to any length.

SOME OF THE AXIOMS

It will probably be highly rewarding if an attempt is made to illustrate these axioms with the help of concrete models. Anything, left to imagination, remains largely vague in case of young students. These axioms are :

1. A straight line joining two points in a plane, produced however far both ways, lies wholly in that plane.
2. Through a given straight line, an infinite number of planes may be made to pass.
3. If a plane of infinite extent rotates about a straight line that lies in it, it may be made to pass through any given point in space outside that line.
4. Three or more straight lines are parallel, when every two of them are parallel.

SCOPE OF SOLID GEOMETRY

In our system, considerable time is spent studying two-dimensional space with very little attention being given to the three-dimensional space in which we live. It seems that solid geometry, a study of space and objects about us, has a practical aspect which should not be neglected, regardless of whether or not the student intends to study higher mathematics. Many mathematics teachers of western countries advocate that essential elements of both the plane and solid geometry be taught in the same year. This viewpoint can be supported by quoting many similarities in the two. These include perpendiculars to lines and planes, parallel lines and planes, congruence and similarity in plane and solid figures, plane and spherical triangles, bisectors, plane and dihedrel angles, circles and spheres, polygons and prisms, loci, areas of plane figures and areas and volumes of solids, and the solid figures generated by moving plane figures through space.

The opinion is also expressed that although a certain development of solid geometry should be carried along with plane geometry, no formal proofs should be required for the solid geometry theorems in such a course. But the proofs shall have to ultimately come at the high school stage.

METHODS TO TEACH

There is no new method apart from the methods already explained in detail. Laboratory work or practical proof will always form the starting point leading to abstract reasoning. When it is to be proved that 'One, and only one, plane may be made to pass through any two intersecting straight lines', two intersecting sticks will represent the intersecting straight lines. One or more than one cardboard pieces will be provided to be used as planes. The students will be given the opportunity to try if they can pass more than one plane through these sticks. They will themselves notice that only one cardboard piece or plane can be made to pass. The theoretical proof of this proposition will further be based on practical demonstration of a plane rotating about one of the two intersecting straight lines. The argument will have to be understood analytically and systematised synthetically. Their combination preceded by practical demonstration will be an effective approach. The imagination of the student is to be exploited while drawing two or more planes on the same sheet of paper and blackboard surface. It is mainly to be taught on the same lines as the demonstrative geometry. The analysis of the statement will have to be done. The student will have to be led from the unknown to the known. For example :

The theorem of three perpendiculars namely,

"If from a given point outside a plane a perpendicular is drawn to the plane, and from the foot of the perpendicular a straight line is drawn ⊥ to a straight line in the plane, the straight line joining the foot of this perpendicular to the given point is perpendicular to the straight line in the plane :

What is given in this statement?

GIVEN

There is a plane XY. P is a point outside this plane. PA is ⊥ to the plane. ML is any straight line in the plane XY, and AB has been drawn ⊥ to XY.

(∵ from the foot of the perpendicular a st. line is drawn ⊥ to a straight in the plane).

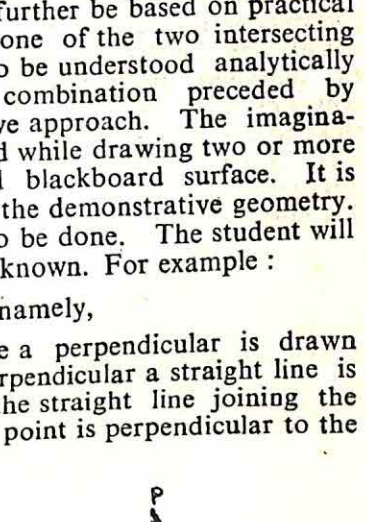


Fig. 88

What is unknown in this statement ?

TO PROVE

The st. line joining the foot of the second perpendicular AB, to the given point P is also $\perp$ to the straight line in the plane, that is PB is $\perp$it ML.

How can a line be proved $\perp$ to another line ?

Either it should be proved that the two adjacent supplementary angles are equal or the Pythagoras theorem may be found applicable.

How to show the equality of two angles ?

The triangles containing them should be proved congruent. But where are the triangles ? Not available, in the diagram. The attempt at construction of two such triangles will not help much.

Now, how to try the Pythagoras Theorem on it ?

A triangle will have to be constructed by joining PM.

$\therefore$ Const :—Join PM.

PB will be $\perp$ to ML
If $PM^2=BM^2+PB^2$.

Can you convert PM^2, BM^2 and PB^2 into some known forms ?

$PM^2=PA^2+AM^2$ (Const :—Join AM).
($\because$ PA is $\perp$ to AM)
$BM^2=AM^2-AB^2$ ($\because$ AB is $\perp$ to ML)
$PB^2=PA^2+AB^2$ ($\because$ PA is $\perp$ to AB).

Substitute these values in the above equation.

$PM^2=BM^2+PB^2$ is true
if $PA^2+AM^2=AM^2-AB^2+PA^2+AB^2$
Which is true.

Hence, retracing the argument :
PB is $\perp$ to ML.

Q.E.D.

PARALLEL PLANES

1. Two planes are said to be parallel when they do not meet however far they are produced. (The roof and the floor of a room are parallel planes.)

2. Three or more planes are parallel when every two of them are parallel. (The floors of a multi-storeyed building are such parallel planes.)
3. When two planes are not parallel, they must intersect when produced, and they do so in a straight line. (The roofs and the floor of a tent are such planes.)

PROPOSITION

Planes to which the same straight line is perpendicular are parallel to one another.

What is given here ?

GIVEN

There are two planes XY and LM. A straight line AB is perpendicular to both of them.

What has to be proved here ?

TO PROVE

Plane XY is ∥ to the plane LM.

How to prove them parallel practically ?

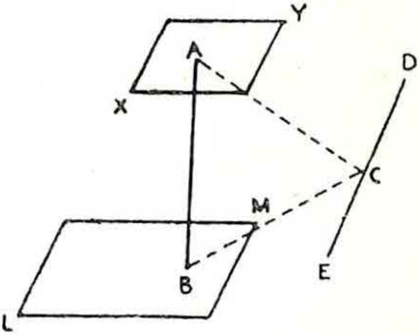


Fig. 89.

The planes may be enlarged by joining more pieces of cardboard. It will be seen that when the planes are produced, they do not come close, remain at the same distance and cannot meet however far they may be produced.

How to prove this theorem by argument ?

We may start with the assumption that the planes are not parallel, and they intersect, when produced, in a straight line. Some construction will have to be made to show that they intersect in a straight line.

CONST

Let them intersect in a straight line DE. How to disprove our assumption or how to show the absurdity of it ?

PROOF

The straight line DE lies in both the planes. If we can show that it is not so, the supposition will not stand.

All the points on DE lie in both the planes.

CONST

Let us take a point C in it and join it with A and B, and see what happens.

PROOF

What will be the angle between AB and BC ?

AB is $\perp$ to BC

Or $\angle ABC = 1$ right angle.

[Since AB is $\perp$ to the plane LM and BC is a straight line in the same plane].

What will be the angle between BA and AC ?

BA is $\perp$ to AC Or $\angle BAC = 1$ right angle.

[Since BA is $\perp$ to the plane XY and AC is a straight line in the same plane].

How can these conclusions be used ?

The sum of $\angle ABC$ and $\angle BAC$ is two right angles, which is impossible, because they are only two angles of the triangle ABC.

$\therefore$ the supposition stands disproved.

Or, the planes do not meet anywhere when produced.

Hence plane XY is $\parallel$ to plane LM.

Q.E.D.

PROJECTION

1. The projection of a point on a plane is the foot of the perpendicular let fall from the point to the plane. (When a bead or tablet is allowed to fall freely on a floor, its point of contact with the floor is its projection and the foot of the perpendicular let fall from the starting point of the bead to the plane).

2. The projection of a line (straight or curved) is the locus of the feet of perpendiculars let fall from the points in that line to the plane. [This can also be shown by throwing freely a straight rod and a curved rod to the floor. The straight rod will touch the plane in a straight line and the curved in a curved one. If we throw beads one by one so that the points of dropping constitute a straight line in one case and a curved line in another, the points of touch with the plane will represent the respective projection of the points of dropping. The loci of the points where the beads touch the plane after

falling perpendicularly will be a straight line and a curved line representing the projections of the corresponding straight and curved lines made by the points of dropping.]

SCOPE OF CORRELATION

Solid geometry can be taught through correlation in many cases. Instances have already been given. A few more instances are given here :

A showcase made of glass has the shape shown in the figure opposite. The top MNRS is parallel to the base ABCD. To cut the inclined face NBCR from a glass plate, the craftsman simply measures the distance NB, draws NR parallel to BC, and then cuts along NR. The inclined face will then lie flat against the top and base. In order to justify the craftman's procedure, we must show that :

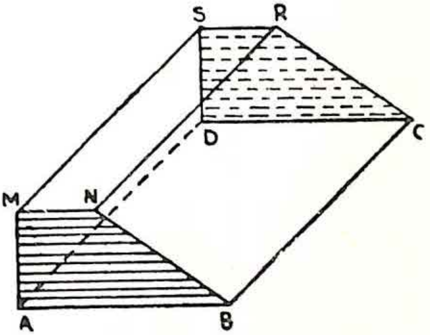


Fig. 90.

If a plane intersects two parallel planes, the intersections are parallel lines.

The top and base of the showcase are assumed to be parallel; that is, plane MR is parallel to plane AC. These two planes are intersected by the plane NC in two straight lines, represented in the figure by the two line segments NR and BC, since we know that the intersection of two planes is a straight line.

Now assume that NR is not parallel to BC. Then, since the line segments NR and BC lie in the same plane NC, they must meet, if extended far enough, at some point P. Then P, being on both line segments NR and BC, would also be in both planes MR and AC. Line segment NR is in plane MR, line segment BC is in plane AC, and a straight line containing any two points in a plane lies wholly in the surface. But two parallel planes have no points in common. Hence, this contradicts the given condition that plane MR is parallel to plane AC. Therefore, the assumption that the line segments are not parallel is wrong, and NR is parallel to BC.

In the construction of a roof truss, such as the one shown in the figure, the purlin (ridgepole) AB is placed parallel to the horizontal timbers CF and DE. The carpenter constructs the framework so that the timbers CF and DE are parallel; then he erects the purlin AB parallel to one of the horizontal timbers and concludes that it must be parallel to the other. His reasoning is based on the fact that if two distinct lines are parallel to the same line, they are parallel to each other.

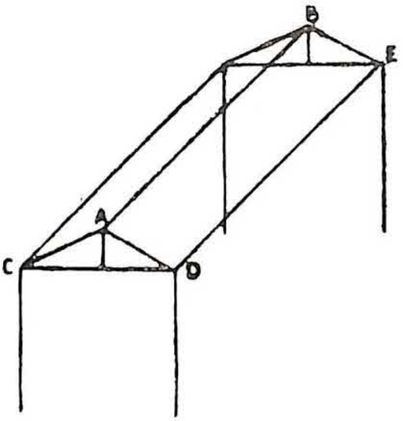


Fig. 91.

A carpenter erects a beam which is to be a support of the roof truss of a cabin. To test whether the beam is perpendicular to the floor of the cabin, he places a carpenter's square with one edge on the floor and the other flat against the beam as in the figure. He then changes the position of the carpenter's square on the floor and again places the other edge flush against the beam. If, in these two positions, the beam each time is in line with the vertical edge of the carpenter's square, the beam is perpendicular to the plane of the floor.

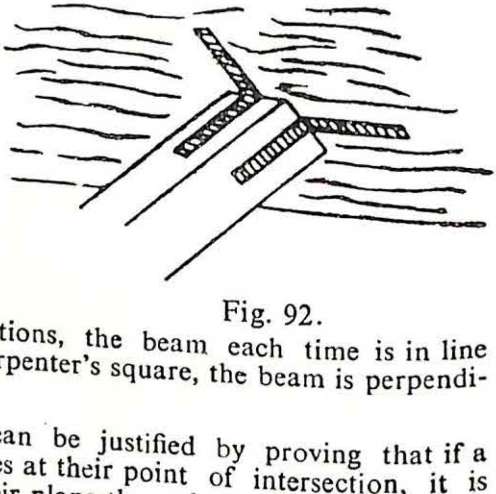


Fig. 92.

The carpenter's conclusion can be justified by proving that if a line is perpendicular to two lines at their point of intersection, it is perpendicular to any line in their plane through that point.

EXERCISES

1. How will you introduce solid geometry to the students who are beginners in this branch ?

2. Explain the procedure which you will adopt to teach the following topics effectively ?

(a) The theorem of three perpendiculars;

(b) Parallel planes.

(c) Projection.

3. What is the scope of the correlation of solid geometry with other branches of mathematics ?

4. How will you make use of illustrations for the teaching of solid geometry ?

CHAPTER TWENTY-TWO

LESSON PLANS

LESSON PLAN—I

CLASS

IX.

SUBJECT

Geometry.

TOPIC

Proposition :—A side of triangle is produced so as to form an exterior angle. Prove that the sum of the opposite angles is equal to this exterior angle.

TIME

40 mts.

APPARATUS

1. Ordinary class-room apparatus.
2. A chart showing three triangles of different shapes with one side produced in each case.
3. Angular pieces of chart paper representing base angles of one of the triangles.
4. Geometry box.

AIMS OF THE LESSON

1. To develop the thinking and reasoning powers of the students.
2. To enable them to prove the proposition in hand.
3. To enable them to apply this proposition to relevant exercises.

PREVIOUS KNOWLEDGE

Students have already proved the proposition that the sum of the angles of a triangle is equal to two right angles. Similarly they have known that a straight angle is always equal to two right angles.

P.K. TESTING AND INTRODUCTION

For this purpose the following questions will be put to them :

(i) If a st. line stands on another st. line, what is the sum of adjacent angles so formed ? [It is equal to two rt. angles.]

(ii) How do you define a straight angle ? [If the two arms of an angle are in the same st. line, the angle is called a st. angle. It is always equal to two rt. angles.]

(iii) What is the sum of supplementary angles ? [It is also equal to two rt. angles.]

(iv) What is the sum of the angles of a triangle ? [It is also equal to two rt. angles or 180 degrees.]

(The chart will be shown here.)

Q:—What do you find in the first figure ?

There is a $\triangle ABC$, whose side BC has been produced to D.

The teacher says, "Let us learn an interesting thing about this figure."

A student will be asked to measure $\angle ABD$.

Another student will be asked to measure $\angle$s A and C of the triangle and to find their sum.

The students will note that the sum of $\angle$s A and C is equal to the angle ABD.

The teacher asks, "How can you state this theorem ?"

[A side of a triangle is produced so as to form an exterior angle. Prove that the sum of opposite angles is equal to the exterior angle.]

STATEMENT OF AIM

The teacher announces, "Today we shall prove the theorem enunciated above".

PRESENTATION

The abovementioned procedure will be followed in case of the remaining two triangles also.

The students will measure the exterior and the interior opposite angles in both the cases and notice through practical measurement that the sum of the interior angles is equal to the exterior angle.

Similarly to give another practical proof of the idea the angular pieces of chart paper will be placed on their equal angles. Then these pieces will be placed in combination on the exterior angle. They will together cover this angle. Therefore it gives another proof.

Now the theoretical proof of the proposition will be given.

Q :—What is given in the problem ?

GIVEN

A △ABC, whose side CB is produced to D, so that an exterior ∠ABD is formed.

(The construction will be performed side by side.)

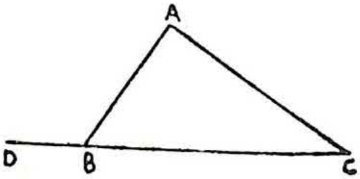


Fig. 93

Q :—What has to be proved in this case ?

TO PROVE

$\angle ABD = \angle BAC + \angle ACB$

That is the exterior angle=sum of the two opposite angles.

PROOF

The teacher asks, "How can you proceed with the proof ?"
(In the same manner as is done in case of practical proofs.)
What is the sum of three angles of a triangle ?
(It is equal to two right angles.)
Write it down in this case :—

$$\angle BAC + \angle ACB + \angle ABC = 2 \text{ rt. } \angle s. \qquad \ldots(i)$$

Find out another group of angles in this diagram whose sum may be equal to 2 rt. ∠s. ?

$\angle ABD + \angle ABC = 2 \text{ rt. } \angle s.$

[∵ AB stands on the st. line CD.] ...(*ii*)

The teacher goes on :—

Where do these two equations (*i*) and (*ii*) lead you to ?

We can say that,

$\angle ABD + \angle ABC = \angle BAC + \angle ACB + \angle ABC$

[$\because$ each of these groups $= 2$ rt. $\angle$s.]

What can you conclude further ?

$\angle ABC$ is common, therefore it can be cancelled on both sides.

$\therefore\ \angle ABD = \angle BAC + \angle ACB$. Q.E.D.

ANOTHER PROOF FOR THE PROPOSITION

Write down 'the given' and to prove' in this case.

GIVEN

A $\triangle ABC$ in which BC is produced to D such that an exterior $\angle ACD$ is formed.

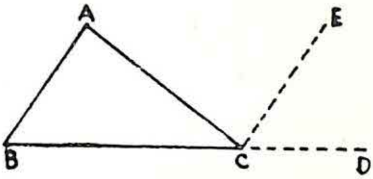


Fig. 94.

TO PROVE

$\angle ACD = \angle ABC + \angle BAC$.

Let us try to divide $\angle ACD$ into two parts so that one part $= \angle ABC$ and the other $= \angle BAC$.

There is a very elegant method for this

CONST.

Through C draw CE is ∥ to BA.

(With the help of cardboard angular pieces, the teacher will show that $\angle BAC = \angle ACE$ and $\angle ABC = \angle DCE$.)

PROOF

What is the reason of the equality of $\angle$s BAC and ACE ?

$\because$ CE is ∥ to BA and AC cuts them

$\therefore\ \angle ACE = \angle BAC$ (alt. $\angle$s) ...(*i*)

What is the reason of the equality of $\angle$s ABC and DCE ?

$\because$ CE is ∥ to BA and BD cuts them

$\therefore\ \angle DCE = \angle ABC$ (corresponding angles) ...(*ii*)

What can you further conclude from (*i*) and (*ii*)

$\angle ACE + \angle DCE = \angle ABC + \angle BAC$

$\therefore\ \angle ACD = \angle ABC + \angle BAC$. Q.E.D.

APPLICATION

(1) In a $\triangle ABC$, BC is produced to D.

If $\angle ACD = 145^\circ$, $\angle A = 85^\circ$.

Find $\angle ABC$ and $\angle ACB$.

What do we know from the proposition ?

$\angle ACD = \angle ABC + \angle BAC$

$\therefore\ 140^\circ = \angle ABC + 85^\circ$

$\therefore\ \angle ABC = 55^\circ$.

What will $\angle ACB$ be equal to then ?

$\angle ACB = 180^\circ - \angle ACD = 180^\circ - 140^\circ = 40^\circ$

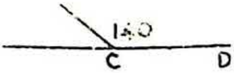


Fig. 95.

RECAPITULATION

The students will be asked to :—

(*a*) enunciate the proposition.

(*b*) draw its diagram.

(*c*) write down the steps of "given" and "to prove" in this case.

HOMEWORK

The students will be asked to reproduce the proposition at home. One exercise will also be given to be proved at home.

LESSON PLAN—II

SUBJECT

Geometry/Algebra/Arithmetic.

CLASS

X.

TOPIC

Identity $a^2 - b^2 = (a+b)(a-b)$

TIME

40 mts.

APPARATUS

1. Ordinary class-room apparatus.
2. A square piece of cardboard.
3. Scissors.
4. A Chart.

AIMS OF THE LESSON

1. To develop thinking and reasoning power of the students.
2. To arouse the interest of the students in the quantitative side of the world.
3. To enable them to understand and prove the identity $a^2-b^2=(a+b)(a-b)$.
4. To enable them to apply this identity to algebraic and arithmetical problems.

PREVIOUS KNOWLEDGE

The students are expected to know addition, subtraction, multiplication and division in algebra and also how to find out areas of square and rectangular figures.

P.K. TESTING AND INTRODUCTION

To test their previous knowledge and to prepare them for the present lesson the following questions will be put to the students.

(*i*) $a \times a=$?
(*ii*) $b \times b=$?
(*iii*) What is the area of a rectangle the length of which is 16′ and the breadth 12′ ?
(*iv*) What is the area of a square one side of which is 10′ ?
(*v*) What is the area of a rectangle with its length a ft. and breadth b ft. ?
(*vi*) What is the area of a square whose one side is a ft. ?
(*vii*) $4^2-3^2=$?
(*viii*) $10^2-8^2=$?
(*ix*) $(101)^2-(99)^2=$?

The students will try to do it by the long process of finding the square of each and then their difference.

STATEMENT OF AIM

The teacher announces, "Today we shall try to find out an easier method of solving such problems with the help of algebra".

PRESENTATION

This stage will be divided and developed in three units.

UNIT I

Recognition of the form a^2-b^2

UNIT II

Tackling the problem with the help of geometry.

UNIT III

Verification through direct multiplication.

UNIT I

The teacher will write the following problems on the blackboard :

(*i*) $(101)^2—(99)^2$
(*ii*) $(197)^2—(3)^2]$
(*iii*) $(217)^2—(83)^2$
(*iv*) $(411)^2—(89)^2$

He will then ask the students to note the similarity of the expressions.

[All are differences of two squares].

The teacher asks, "How will you give them a general form ?"

[By writing a^2 as the first square and b^2 as the second square.]

He goes on, "Then our problem reduces itself to finding out the general form of the problems in hand. So let us now solve $a^2—b^2$.

UNIT II

(By showing the square piece of cardboard, whose side has been labelled as '*a*'.)

Q. :—What is the area of this square ABCD ?

(Square ABCD$=a^2$)

Q. :—What is the area of the square AEFG ?

(Square AEFG$=b^2$)

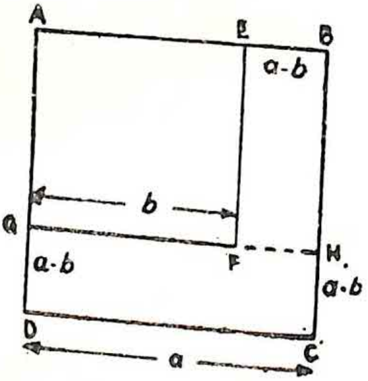


Fig. 96.

Q. :—Then, what is the area of the remaining part of the cardboard ? (The teacher will here remove the AEFG portion of the cardboard by cutting it with scissors. The remaining portion will thus be EBCDGF. It's area$=a^2—b^2$).

The teacher will further cut the portion BHFE, and place it in continuation with the remaining rectangle GDCH so that BE coincides with CH.

Q. :—What is the shape of the resulting figure ?
(Rectangular)

Q. :—What is the length of this rectangle DKLG ?
$(a+b)$

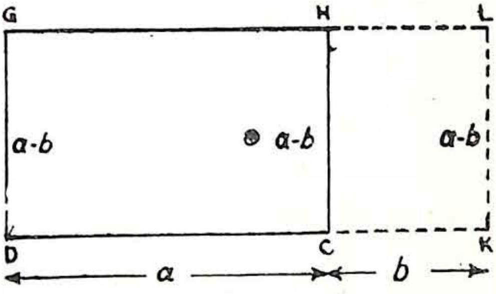


Fig. 97.

Q. :—What is the breadth ? $(a-b)$

Q. :—What is the area ? $(a+b)(a-b)$

Q. :—So what do you conclude from this ?

$$a^2-b^2=(a+b)(a-b).$$

The teacher will then show the chart, which represents x^2-y^2, and by the process of questioning on the basis of two diagrams similar to the above as obtained from the cardboard, will lead the students to conclude that

$$x^2-y^2=(x+y)(x-y)$$

UNIT III

The teacher will ask the boys to actually multiply $(a+b)(a-b)$, $(x+y)(x-y)$ and see what the results are. Similarly they will be asked to actually multiply in a few more cases. The results will be written on the blackboard.

$$(a+b)(a-b)=a^2-b^2.$$
$$(x+y)(x-y)=x^2-y^2.$$
$$(c+d)(c-d)=c^2-d^2.$$
$$(l+m)(l-m)=l^2-m^2.$$

The results will then be generalised with the active participation of the students.

GENERALISATION

Q. :– What is the similarity in the terms on the R.H.S. in all results ? (They are the differences of the squares of two quantities.)

Q. :— What is the similarity in the terms on the L.H.S. in all results ? (They are the products of the sum and difference of the respective quantities.)

Therefore :—

The difference of the squares of two quantities is equal to the product of their sum and difference.

APPLICATION

The teacher will give the following problems for application :—

(*i*) Solve $9x^2-y^2$.

He will put questions to the students and proceed in the following manner :—

$9x^2=(3x)^2$ and $y^2=(y)^2$

Their sum $(3x+y)$ and difference $3x-y$

So $9x^2-y^2=(3x+y)(3x-y)$

(*ii*) Solve $(101)^2-(99)^2$.

What is their sum ?	$101+99=200$
What is their difference ?	$101-99=2$
What is their product ?	$200\times 2=400$
	Hence etc. etc.

The following problems will be solved by the students independently :—

(*i*) a^2-9.
(*ii*) $(197)^2-(3)^2$.
(*iii*) $(411)^2-(89)^2$.
(*iv*) $(217)^2-(83)^2$.

RECAPITULATION

It will just serve the purpose of a quick and short over-all revision. It will be done with the help of the following questions :—

1. What is the difference of the squares of two quantities equal to ?
2. What is m^2-n^2 equal to ?
3. What is $(91)^2-(9)^2$ equal to ?

HOMEWORK

Five problems of the relevant exercise of their textbook will be set as homework.

LESSON PLAN III

SUBJECT

Arithmetic.

TOPIC

Volume of a cuboid.

CLASS

X.

TIME

40 minutes.

APPARATUS

1. Ordinary class-room apparatus.
2. One inch cube piece of wood or any solid material.
3. A chart showing diagrams of square and rectangle.
4. Two cuboids of soap with different dimensions.
5. A tin.
6. A graduated cylinder.
7. One centimetre cube piece of some solid.

AIMS OF THE LESSON

1. To develop thinking and reasoning powers of the students.
2. To enable them to understand the concept of volume.
3. To enable them to find the formula of the volume of a cuboid.
4. To enable them to apply that formula to relevant problems.

PREVIOUS KNOWLEDGE

The students are expected to know the areas of square and rectangle.

P.K. TESTING AND INTRODUCTION

To test their previous knowledge and to prepare them for the new lessons, the following questions will be put to them :—

1. (By showing the diagram of a square on the chart.) How do you calculate the area of this diagram ?
   [Area of a square = Side × Side]
2. (By showing the diagram of a rectangle.) How will you calculate the area of this diagram ?
   [Area of a rectangle = Length × Breadth].
3. (By showing the tin.) If you have to say that this tin can

contains so much oil, how can you express ? [By giving the space occupied by the tin.]

4. Similarly, if you want to say that a tank can contains so much of water, how can you express ? [By telling the space of the tank.]
5. What is the shape of a tin or a tank ? [It is cuboid.]
6. [By showing the cuboid]. How would you name this solid ? [This is a cuboid]
7. How will you find out the space or volume of this cuboid ?

STATEMENT OF THE AIM

The students will not be able to answer the last question and the teacher will announce, "Today we shall find out the method of calculating the volume of a cuboid". The aim will be simultaneously written on the blackboard.

PRESENTATION:—

METHOD	MATTER
1. (By showing the 1 cm. cube) What are the dimensions of this solid ?	It is one cm. long, broad or thick.
2. What is its shape ?	It is known as a cube. The cube has all the three dimensions uniform, whereas in the cuboid these may be different.
3. What is the difference between a cube and a cuboid ?	
4. What is the volume of this cube ?	The teacher will say, "Let us measure it with the help of a graduated cylinder containing water". The initial reading of the water level will be taken and then the solid will be immersed. The water will rise by one cubic centimetre.
5. (By showing the 1 inch cube.) What are the dimensions of this cube ?	Length = 1″ Breadth = 1″ Thickness or Height = 1″
6. Then, what is the volume of this cube ?	It is one cubic inch. { These, one cubic centimetre and one cubic inch, are taken as the units of volume. }
7. (By showing the cuboid of soap)	Length = 5″ Breadth = 3″

What are the dimensions of this cuboid ?	Thickness or Height = 2″ . . . (*i*)
8. In how many parts, have its length, breadth and thickness respectively been divided ?	The length is divided into 5 parts. The breadth is divided into 3 parts. The thickness is divided into 2 parts.
{The teacher will promptly cut this cuboid along the lines of division. It will result into 30 parts.}	
9. What is the shape of each smallpart ?	It is a cube.
10. What are the dimensions of this small cube ?	Length = 1″ Breadth = 1″ Thickness = 1″
11. What is the volume of this small cube ?	It is one cubic inch. This cube will be compared with its equal already shown to the students.
12. Then, what is the volume of the whole cuboid ?	The volume of the cuboid = 30 cubic inches . . . (*ii*)
13. (By showing the second cuboid of soap.) What are the dimensions of this cuboid ? The teacher will cut them quickly in the same manner along the lines of division. It will result in 48 small parts.	Length = 6″ Breadth = 4″ Thickness or Height = 2″ . . . (*iii*)
14. What is the volume of one small cube ?	It is one cubic inch.
15. How many such cubes are there in all ?	There are 48 such cubes.
16. Then, what is the volume of the whole cuboid ?	The volume of the cuboid— 48 cubic inches . . . (*iv*)

GENERALISATION

For the purpose of generalisation, the teacher will draw the students' attention to the pairs (*i*), (*ii*) and (*iii*), (*iv*), and will ask them to observe these to find out some relationship between the dimensions and the volumes of cuboids.

1. What are the dimensions in the first case ?	L=5″ B=3″
2. What is the volume in this case ?	H or T=2″ =30 cubic inches. Volume
3. What are the dimensions in the second case ?	L=6″ B=4″
4. What is volume in this case ?	H or T=2″Volume=48 cubic inches.
5. What is the relation between the volume and dimensions ?	Volume is the product of these three dimensions.

Therefore volume of a cuboid = Length × Breadth × Thickness or Height.

or L × B × T Or H.

APPLICATION

Q. 1. What is the volume of air in your class room, if its dimensions are as follows:—

Length = 25 ft.
Breadth = 20 ft.
Height = 15 ft.

CORRELATION

What is the use of finding out the volume of air in a room ? [While sitting in a room the person should get a regular supply of fresh air. The minimum essential volume of air should be available to everybody. With the help of total volume of air in a room, a lecture hall or a cinema hall, we can fix its comfortable seating capacity. If we try to accommodate more than this fixed number, there will be uneasiness and suffocation for every body].

Q. 2. Find the volume of water in a tank if its dimensions are as follows:—

Length = 150 ft.
Breadth = 80 ft
Depth = 12 ft.

CORRELATION

What is the use of calculating the volume of water in a tank ? [There are certain regions where rain water is stored in tanks for later use for irrigation. The volume of stored water will enable us to know, how much area will it irrigate and for how long.]

RECAPITULATION

1. What do you mean by volume ?
2. What is the difference between a cube and a cuboid ?
3. What is the formula for the volume of cuboid ?
4. What is the need of finding out volume in different cases ?

HOMEWORK

LESSON PLAN—IV

CLASS : IX
SUBJECT : Arithmetic
TOPIC : Compound Interest
TIME : 40 mts.

APPARATUS :

1. Ordinary classroom apparatus.
2. A chart showing a person depositing money in the bank, and withdrawing it after five years along with interest.
3. The chart showing different rates of interest as given by post office, private banks and govt. bank on different kinds of deposits.

AIMS OF THE LESSON

1. To develop the thinking and reasoning powers of the students.
2. To enable them to find out and understand the formula for calculating compound interest.
3. To enable them to apply that formula to relevant problems.

PREVIOUS KNOWLEDGE

The students are expected to know simple interest. They also know that in simple interest.

$$\text{Amount} = P\left(1+\frac{r\times t}{100}\right)$$

P.K. TESTING AND INTRODUCTION

To test their previous knowledge and to prepare them for the new lesson, the following questions will be put to them :

Q.—1. Have you seen a bank ? (Yes).

Q.—2. What is it meant for ? (We can deposit money in it and also we can borrow money from it.)

Q.—3. When you deposit a sum, what are you paid for it ? (Interest) (The relevant chart as mentioned in the apparatus will also be shown.)

Q.—4. What type of interest do you get from a bank ?

ANNOUNCEMENT OF THE AIM

The students are not likely to answer this question correctly. The teacher will announce that the banks pay an interest, different from simple interest, known as Compound interest. He will say, "We shall try to understand it after distinguishing it from Simple interest and solve some of its problems after finding out its formula.

PRESENTATION

The teacher goes on, "Suppose you deposit a sum of Rs 100=00, which pays an interest at 5%, by how much will your deposit increase, when the interest is added to it after a year ?

[The deposit will increase by five rupees and it will become Rs 105=00]

METHOD	MATTER
1. On which amount will the interest for the second year be calculated ?	It will be calculated on Rs 105=0.
2. What will be the interest ?	$\overset{21}{\cancel{105}} \times \dfrac{5}{\underset{\underset{4}{\cancel{20}}}{\cancel{100}}} = \text{Rs } 5\text{—}25 \text{ nP.}$
3. What will be the amount in the name of the depositor after the second year ?	Rs 105+Rs 5.25=Rs 110.25
4. On which amount will the interest for the third year be calculated ?	It will be calculated on Rs 110.25

5. Then, what is the difference between the simple and compound interests ?	In simple interest, the interest is calculated on the original amount for all the years. Whereas in compound interest the depositor gets interest on interest also. That is, the principal for every year is the sum of the original principal and the compound interest of previous years. The amount of previous year becomes principal for the next year.
6. Let us try to understand how to calculate compound interest. What is the formula for calculating amount ?	Amount $=P\left(1+\frac{R\times T}{100}\right)$
7. Take the problem in hand. What will be the amount at the end of the first year ?	$A_1=100\left(1+\frac{5}{100}\right)$ (i)
8. What is the Principal for the second year ?	Principal $=[100(1+\frac{5}{100})]$
9. What will be the amount at the end of second year ?	$A_2=[100(1+\frac{5}{100})(1+\frac{5}{100})$ or $A_2=100(1+\frac{5}{100})(1+\frac{5}{100})$ $=100(1+\frac{5}{100})^2$ (ii)
10. What is the Principal for the third year ?	Principal $=[100(1+\frac{5}{100})(1+\frac{5}{100})$
11. What will be the amount at the end of the third year ?	$A_3=[100(1+\frac{5}{100})(1+\frac{5}{100})](1+\frac{5}{100})$ or $A_3-100(1+\frac{5}{100})(1+\frac{5}{100})(1+\frac{5}{100})$ or $A_2=100(1+\frac{5}{100})^3$ (iii)
12. Observe (i), (ii), (iii) and give the amount at the end of the fourth year ?	$A_4=100(1+\frac{5}{100})(1+\frac{5}{100})(1+\frac{5}{100})(1+\frac{5}{100})$ or $A_4=100(1+\frac{5}{100})^4$ (iv)
GENERALISATION	
13. Observe (i), (ii) and (iii) and (iv) and give the amount after *n* (any number) years.	$A_n=100(1+\frac{5}{100})(1+\frac{5}{100})$ . . . *n times* or $A_n=100(1+\frac{5}{100})^n$
14. Generalise it, so that it may be applicable to all such problems.	Amount $=P\left(1+\frac{R}{100}\right)^n$ after *n* years
15. What will be the formula for compound interest then ?	Compound interest $=P\left(1+\frac{R}{100}\right)^n-P.$

APPLICATION

For the purpose of application, the teacher will solve the following problem on the B.B.

Q.—Find out compound interest on Rs 800.00 for three years at 5% per annum ?

METHOD	MATTER
1. What is the formula for compound interest ?	$\text{C. I} = \left(1+\frac{R}{100}\right)^{n} - P.$
2- Make proper substitutions in the formula ?	$\text{C. I.} = 800\left(1+\frac{5}{100}\right)^{3} - 800$
3. Give the next step ?	$=\overset{\cancel{2}}{\cancel{800}}\times\frac{\overset{21}{\cancel{105}}}{\cancel{100}}\times\frac{\overset{21}{\cancel{105}}}{\underset{\underset{\cancel{1}}{\cancel{20}}}{\cancel{100}}}\times\frac{\overset{21}{\cancel{105}}}{\underset{\underset{10}{\cancel{20}}}{\cancel{100}}}-800$
4. What is the next step ?	$=\frac{9261}{10}-800=\frac{1261}{10}$
5. What is the answer ?	Rs 126.10

For further practice and application the following two questions will also be given to be solved —

1. Find out compound interest on Rs 1250.00 for three years at 4% per annum ?
2. Find out compound interest on Rs 2000.00 for four years at $2\frac{1}{2}\%$ per annum ?

CORRELATION

The teacher will show the chart showing different rates of interest as given by different depositing agencies on different types of accounts.

The students will be asked to point out the best kind of deposit and the best agency.

RECAPITULATION

It will be done with the following questions :

1. What is the difference between S.I. and C.I. ?
2. Why do banks pay C.I. on deposits ?
3. What is the formula for calculating C.I. ?

HOME WORK

A few questions framed on the basis of the rates of interest shown on the chart.

LESSON PLAN V

CLASS

XI

SUBJECT

Trigonometry

TOPIC

To prove geometrically that

$$\sin (A+B) = \sin A \cos B + \cos A \sin B$$

TIME

40 minutes

MATERIAL AIDS

1. Ordinary class room equipment
2. A chart showing $\sin \theta = \frac{MP}{OP}$ and $\cos \theta = \frac{OM}{OP}$
3. A chart showing the figure necessary for proving the formula in hand.

AIMS OF THE LESSON

1. To develop the thinking and reasoning powers of the students.
2. To enable them to prove the above mentioned formula.
3. To enable them to apply the formula to relevant problems.

PREVIOUS KNOWLEDGE

The students are expected to know the values of $\sin \theta$ and $\cos \theta$.

P.K. TESTING AND INTRODUCTION

To test their previous knowledge and to prepare them for the new lesson, following questions will be put to them :

Q. 1. (Showing the figure)
What is $\sin \theta$ equal to ?

$$\left(\sin \theta = \frac{MP}{OP} \right)$$

Q. 2. What is $\cos \theta$ equal to?

$$\left(\cos \theta = \frac{OM}{OP} \right)$$

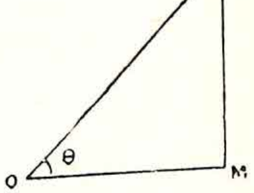


Fig. 98

Q. 3. (Drawing the figure)
What is ∠ZOX equal to ?
(∠ZOX = A+B)

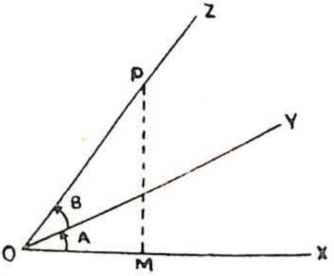


Fig. 99

Q. 4. (After drawing PM ⊥ OX)
What is sin (A+B) equal to ?

$$\left[\sin (A+B) = \frac{MP}{OP}.\right]$$

Q. 5. How will you express sin of compound angle A+B in terms of trigonometrical ratios of angles A and B ?

ANNOUNCEMENT OF THE AIM

Finding the students unable to answer the last question, the teacher will announce the aim of the lesson by saying that today we shall learn how to express sin (A+B) in terms of trigonometrical ratios of angles A and B, or to prove that sin (A+B) = sin A cos B + cos A sin B.

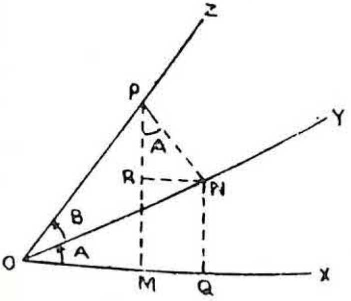
PRESENTATION

The teacher will present the lesson and develop it through the following questions and answers:

Q. How will you make constructions in the above figure so that you may be able to write the trigonometric ratios of angles A and B ?

(Draw PN ⊥ OY. Draw NQ ⊥ OX and draw NR ⊥ PM)

Q. What is the value of
∠NPR ?
(∠NPR = A
∵ ∠NPR = 90°− ∠RNP
= ∠ONR = ∠XOY)



Q. What is sin (A+B) equal to ?

Fig. 100

$$\left[\ \sin(A+B) = \frac{MP}{OP}\ \right]$$

Q. What is $\frac{MP}{OP}$ equal to ?

$$\left[\ \frac{MP}{OP} = \frac{MR+RP}{OP} = \frac{QN+RP}{OP} = \frac{QN}{OP} + \frac{RP}{OP}\ \right]$$

Q. How to transform $\frac{QN}{OP}$ to incorporate the trigonometric ratios of angles A and B ?

$$\left[\ \frac{QN}{OP} = \frac{QN}{ON} \cdot \frac{ON}{OP}\ \right]$$

Q. How to transform $\frac{RP}{OP}$?

$$\left[\ \frac{RP}{OP} = \frac{RP}{NP} \cdot \frac{NP}{OP}\ \right]$$

Q. What is the next step to express sin (A+B) in terms of the trigonometric ratios of angles A and B ?

$$\left[\ \begin{aligned} \sin(A+B) &= \frac{MP}{OP} = \frac{QN}{OP} + \frac{RP}{OP} \\ &= \frac{QN}{ON} \cdot \frac{ON}{OP} + \frac{RP}{NP} \cdot \frac{NP}{OP} \end{aligned}\ \right]$$

Q. What is $\frac{QN}{ON}$ equal to ? $\left(\frac{QN}{ON} = \sin A\right)$

Q. What is $\frac{ON}{OP}$ equal to ? $\left(\frac{ON}{OP} = \cos B\right)$

Q. What is $\frac{RP}{NP}$ equal to ? $\left(\frac{RP}{NP} = \cos A\right)$

Q. What is $\frac{NP}{OP}$ equal to ? $\left(\frac{NP}{OP} = \sin B)\right)$

GENERALISATION

Q. Then what is sin (A+B) equal to ?
[sin (A+B) = sin A cos B + cos A sin B]

APPLICATION

The following problems will be given for application:

1. Apply the formula to sin (C+D).

2. Find the value of $\sin 75^\circ$.
3. Prove that $\sin 19^\circ \cos 11^\circ + \cos 19^\circ \sin 11^\circ = \frac{1}{2}$.

Note : The teacher will transfer the essential on the material on the black board and ask the students to note it down side by side]

RECAPITULATION

1. What is $\sin (A+B)$ equal to ?
2. Draw the figure employed in this proof.
3. What is $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$ equal to ?

HOME WORK

The students will be asked to solve at home five relevant problems from their text book.

EXERCISES

1. Prepare a lesson plan on any one of the following topics :
   (a) Identity $K(a+b+c) = Ka+Kb+Kc$
   (b) Area of a circle.
   (c) Angle—sum of a triangle.
2. Draw up a list of steps which should be observed while preparing a lesson plan in mathematics. Give reasons for each and every step ?
3. Give the lesson plan of any topic actually taught by you to the students.
4. What points should be kept in mind while observing and criticising a lesson in mathematics ?

BIBLIOGRAPHY

Adams, L.D., *A Background of Primary School Mathematics.*

Aiyanger, N.K., *The Teaching of Mathematics in the New Education.*

Association of Assistant Masters in Secondary Schools of U.K., *The Teaching of Mathematics.*

Ball, W.W. Rouse, *Mathematical Recreations and Essays.*

Ballard, P.B., *Teaching of Essentials of Arithmatic.*

Bell, E.T., *The Development of Mathematics.*

Bhatia and Bhatia, *The Principles and Methods of Teaching.*

Breslich, E.R., *Problems in Teaching Secondary School Mathematics.*

Brideoake and Groves, *Arithmetic in Action.*

Bryan, Thwartes, *On Teaching Mathematics.*

Butler and Wren, *The Teaching of Secondary Mathematics.*

Cajori, F., *A History of Mathematics.*

Chadha, B.N., *The Teaching of Mathematics.*

Cutto Leonard, *Teaching Yourself Mathematics.*

Davis R. Davis, *The Teaching of Mathematics.*

Dharam Vir and V.N., Aggarwal, *The Teaching of Mathematics in India.*

Dudeney, Henry E., *Amusements in Mathematics.*

Freebury, H.A., *A History of Mathematics.*

Godfre, & Siddons, *The Teaching of Elementary Mathematics.*

Hickerson, J. Allen, *Guiding Children's Arithmatical Experiences.*

Hogben, Lancelot, *Mathematics for the Million.*

Hooper, Alfred, *Makers of Mathematics.*

Kapur, J.N., *The Spirit of Mathematics.*

Lamborn, *Reason in Arithmatic.*

Lee, Doris M., *A Background to Mathematical Development.*

Marks, Pudly, Kinney, *Teaching Arithmatic for Understanding.*

Mcswain & Cooke, *Understanding and Teaching Arithmatic.*

Mathematical Association London—A report, *Teaching of Trigonometry in Schools.*

Ministry of Commerce & Industry, Govt. of India, *Teaching of Metric System.*

Mira, Julio, A., *Geometry through Practical Applications.*

Murdoch, *Mathematics in Post-Primary Schools.*

National Council of Teachers of Mathematics—Washington, *The Growth of Mathematical Ideas.*

Potter, F.F., *The Teaching of Arithmetic.*

Russell, B., *Introduction to Mathematical Philosophy.*

Salamat Ullah, Dr *The Basic Way to Arithmetic.*

Sanford, Vera, *A Short History of Mathematics.*

Schultze, A., *The Teaching of Mathematics in Secondary Schools.*

Shaw & Wright, *Discovering Mathematics.*

Singh, D.P., *Correlation in Basic Education.*

Smeltzer, Donald, *Man and Number.*

Spencer and Brydegard, *Building Mathematical Concepts in the Elementary School.*

Spitzer, *The Teaching of Arithmetic.*

Smith, *A History of Mathematics.*

Smith & Lovell, *The Teaching of Arithmetic in Primary Schools.*

Smith D.E. & W.D. Reeve, *The Teaching of Junior High School Mathematics.*

Summer, W.L., *The Teaching of Arithmetic and Elementary Mathematics.*

Trimble Hamilton, Silvey, *Basic Mathematics for General Education.*

Verma, K.K., *A First Course in Teacher Education.*

Westaway, F.W., *Craftsmanship in the Teaching of Elementary Mathematics.*

Wilson, Guy, M., *Teaching of New Arithmetic.*

Young, J.W.A., *Teaching of Mathematics.*

S.C.E.R.T. West Bengal
Library
Calcutta
25/3, B. C.

Dr Kulbir Singh Sidhu, M.A. (Maths, Psy.) M.Ed. (Gold Medallist), Ph.D., P.E.S., is Principal, Montgomery Guru Nanak College of Education, Jalandhar. A prolific writer, he has to his credit many books including, 'The Statistics in Education and Psychology', 'School Organisations and Administration' *and* 'Methodology of Research in Education'.

372.7
SID

Released Recently

Methodology of Research in Education
Dr. Kulbir Singh Sidhu

Research in every field and more so in the field of education is demand of the day. Progress in any field is directly linked with research in that field. Our problems and difficulties in the field of education further necessitate a purposeful and sustained research effort.

Educational planning has to be based on sound research findings. The researches carried out at other places may also help us in many ways but we cannot always depend on the borrowed and imported stuff.

The educational programmes of modern India cannot be run on age-old lines. So long as our research effort in Education remains poor we shall continue to be counted as an educationally backward nation. Educational reform and progress necessitate a dedicated and competent team of research scholars.

The present book is being brought out to give to scholars another source. It may be found to have enriched the existing literature by putting most of the relevant material at one place and by providing to the needy student and teacher a compact volume on the subject. The readers will find the language and style of the book readable and acceptable.

Dr Kulbir Singh Sidhu, M.A. (Maths, Psy.), M. Ed. (Gold Medallist), Ph.D, P.E.S., is Principal, Montgomery Guru Nanak College of Education, Jalandhar. The author has taught Methodology and Research in Education for about two decades. His background and experience in this subject have prompted him to produce this book. A prolific writer, he has to his credit many books including *The Statistics in Education and Psychology*, and *School Administration*.

STERLING PUBLISHERS PRIVATE LIMITED
New Delhi-110016
Bangalore-560001